A

KEY TO THE EXERCISES

IN THE

FIRST SIX BOOKS

OF

CASEY'S ELEMENTS OF EUCLID.

SECOND EDITION.

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CASEY'S ELEMENTS OF EUCLID.

BY

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EXERCISES ON EUCLID.

BOOK I.

PROPOSITION I.

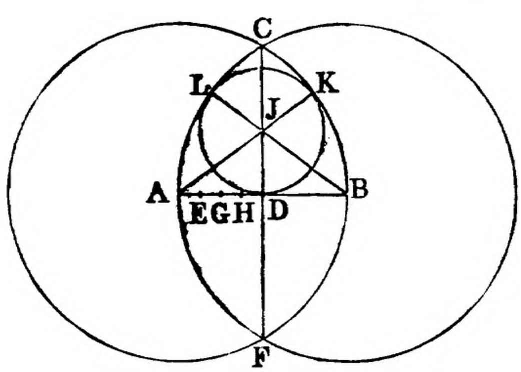
1. **Dem.**—The four lines AC, AF, BC, BF are each = AB, and $\therefore$ = to each other. Hence ACBF is a lozenge.

2. **Dem.**—Because AC = BC, and CF common, and the base AF = BF; $\therefore$ (VIII.) the $\angle$ ACF = BCF; $\therefore$ ACF is $\frac{1}{2}$ an $\angle$ of an equilateral Δ. Again, the $\angle$ CAB = ACD + ADC (XXXII.); but ACD = ADC; $\therefore$ CAB = 2ACD; $\therefore$ ACD is $\frac{1}{2}$ an $\angle$ of an equilateral Δ, and ACF is $\frac{1}{2}$ an $\angle$ of an equilateral Δ; $\therefore$ DCF is an $\angle$ of an equilateral Δ. Similarly DFC is an $\angle$ of an equilateral Δ. Hence the Δ CDF is equilateral.

3. **Dem.**—Join AF. Because AG = AF, the $\angle$ AGF = AFG; and because AF = AC, the $\angle$ ACF = AFC; $\therefore$ the $\angle$ GFC = FGC + FCG, and is $\therefore$ (XXXII. *Cor.* 7) a right $\angle$. In like manner HFC is a right $\angle$. Hence (XIV.) G, F, H are collinear.

4. **Dem.**—$GC^2 = GF^2 + FC^2$ (XLVII.), and $GC^2 = 4AG^2$; $\therefore GF^2 + FC^2 = 4\,AG^2$; but GF = AG. Therefore $FC^2 = 3AG^2 = 3AB^2$.

5. **Sol.**—Join CF. Divide AD into four equal parts in E, G, H.



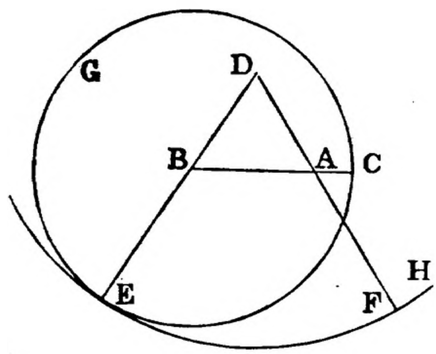
From DC cut off DJ = ED. J is the centre of the required $\odot$.

Dem.—Join AJ, BJ, and produce them to meet the $\odot$s in K, L.

Because the ∠ ADJ is right, $AJ^2 = JD^2 + DA^2 = 3^2 + 4^2 = 5^2$; ∴ AJ is = 5 of the parts into which AD is divided; but AK = AB; ∴ JK = 3 of the parts; ∴ JK = JD. Again, AD = DB, and DJ common, and the ∠ ADJ equal BDJ; ∴ (IV.) AJ = BJ; but AK = BL; ∴ JK = JL. Hence the lines JD, JK, JL are equal; and the ⊙, with J as centre and JD as radius, will pass through the points K, L.

PROPOSITION II.

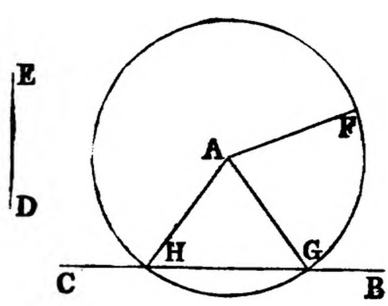
1. **Sol.**—On AB describe the equilateral △ ABD. With B as centre and BC as radius, describe the ⊙ CEG, and produce DB



to meet it in E. With D as centre and DE as radius, describe the ⊙ EFH, and produce DA to meet it in F. AF is the required line.

Dem.—Because D is the centre of the ⊙ EFH, ∴ DE = DF; but DB = DA; ∴ BE = AF, and BE = BC; ∴ AF = BC.

2. **Sol.**—Let A be the given point, and BC the given line.



It is required from the point A to inflect to BC a line equal to a given line DE. From A draw AF = DE [II.]. With A as centre,

and AF as radius, describe a ⊙ cutting BC in G, H. Join AG, AH. AG, AH are the required lines.

Dem.—Because AF = AG, and AF = DE; ∴ AG = DE. In like manner AH = DE. Hence there are two solutions.

PROPOSITION IV.

1. Let AD bisect the vertical ∠ of the isosceles △ ABC. It is required to prove that it bisects the base BC perpendicularly.

Dem.—AB = AC, and AD common, and the ∠ BAD = CAD ∴ (IV.) the ∠ ADB = ADC, and the side BD = CD. Hence BC is bisected, and (Def. XIV.) AD is ⊥ to BC.

2. **Dem.**—Let ABCD be the quadrilateral, and BD its diagonal. Because AB = CB, and BD common, and the ∠ ABD = CBD; ∴ (IV.) the base AD = CD.

3. Let the lines AB, CD, bisect each other in E.

Dem.—Take any point F in ED. Join AF, BF. Because AE = BE, and EF common, and the ∠ AEF = BEF; ∴ the base AF = BF.

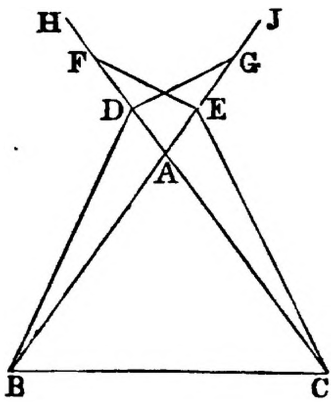
4. Let ABC be the △. On the sides AB, AC, describe equilateral △s ABD, ACE. Join CD, BE. It is required to prove that CD = BE.

Dem.—Because the ∠ DAB = CAE, to each add the ∠ BAC; then the ∠ DAC = BAE; and since DA = BA, and CA = EA, the sides DA, AC = BA, AE, and we have shown that the ∠ DAC = BAE; ∴ (IV.) the bases CD, BE, are equal.

PROPOSITION V.

1. (1) **Dem.**—Take any point D in AB, and from AC cut off AE = AD (III). Join BE, CD, DE. Because AB = AC, and AE = AD; ∴ BA and AE = CA and AD, and the ∠ A is common; ∴ BE = CD, and the ∠ ABE = ACD. Again, because BE = CD, and BD = CE; ∴ BD and BE = CE and CD, and the ∠ DBE = ECD; ∴ (IV.) the ∠ BDE = CED, and the ∠ BED = CDE; hence the remainders, the ∠s BDC, BEC, are equal. Again, BD = CE, and DC = EB; ∴ BD and DC = CE and EB, and the contained ∠s BDC, CEB, have been shown to be equal; ∴ (IV.) the ∠s DBC, ECB, are equal.

(2) **Dem.**—Produce BA, CA, to J, H, in AJ; take any points E, G, and from AH cut off AD = AE, and AF = AG. Join DG,

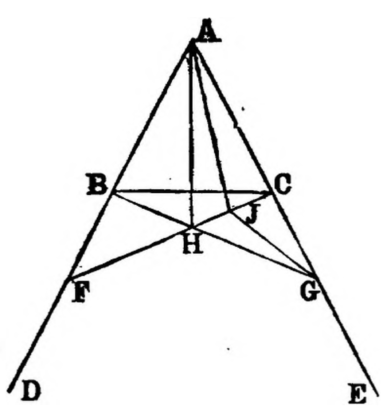


DB, EC, EF. Because AF = AG, and AE = AD; ∴ AF and AE = AG and AD, and the ∠ FAG common; ∴ the base FE = DG, and the ∠ AFE = AGD, and the ∠ FEA = GDA.

Again, because BG = CF, and GD = FE; ∴ BG and GD = CF and FE, and the ∠ DGB = EFC; ∴ the base DB = EC, and the ∠ GDB = FEC; but the ∠ GDA = FEA; ∴ the remainders, the ∠s BDC, BEC, are equal.

Now, since BD = CE, and DC = EB; ∴ BD and DC = CE and EB, and the ∠ BDC = CEB; ∴ the ∠ DCB = EBC.

2. **Dem.**—If AH be not an axis of symmetry, let AJ be one. Join JG. Because AF = AG, and AJ common, and the ∠ FAJ



GAJ (hyp.); ∴ the ∠ AFJ = AGJ; but the ∠ AFC = AGB; ∴ the ∠ AGJ = AGB, a part = to the whole, which is absurd; ∴ AH must be an axis of symmetry.

3. Let ABC, DBC, be the two isosceles Δ^s on the same base. Join their vertices A, D.

Dem.—The $\angle$ ABC = ACB (v.), and the $\angle$ DBC = DCB (v.); $\therefore$ the $\angle$ ABD = ACD. Now, the two Δ^s ABD, ACD, have the two sides AB, BD = the two sides AC, CD, and the contained $\angle^s$ ABD, ACD, equal; $\therefore$ (IV.) the $\angle$ BAD = CAD. Hence AD is an axis of symmetry.

4. Let AD be the bisector of the $\angle$ BAC.

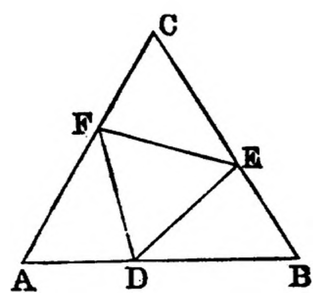
Dem.—Because BA and AD = CA and AD, and the $\angle$ BAD = CAD (hyp.); $\therefore$ (IV.) the $\angle$ ABD = ACD.

5. Let ABCD be the lozenge, and AD, BC, its diagonals.

Dem.—Because AB = AC, the $\angle$ ACB = ABC, and because DB = DC, the $\angle$ DCB = DBC; $\therefore$ the $\angle$ ACD = ABD. Now, the Δ^s ACD, ABD, have two sides AC, CD, and the contained $\angle$ ACD, equal to the sides AB, BD, and the contained $\angle$ ABD; $\therefore$ (IV.) the $\angle$ CAD = BAD, and the $\angle$ CDA = BDA. Hence AD is an axis of symmetry.

6. Let ABC be the Δ.

Dem.—Take three points D, E, F, in the sides AB, BC, CA, equally distant from the vertices A, B, C. Join DE, EF, FD. It is required to prove that the Δ DEF is equilateral. Evidently



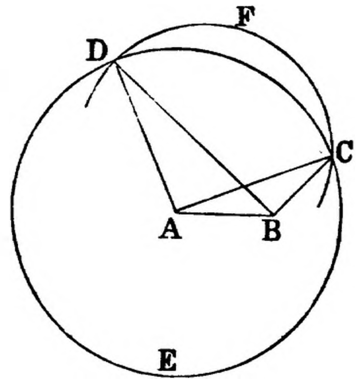
from the given conditions the Δ^s BDE, CEF, AFD, are equal; $\therefore$ their bases DE, EF, FD, are equal. Hence the Δ DEF is equilateral.

PROPOSITION VII.

3. If possible let two $\odot^s$ whose centres are A, B, intersect in the points C, D, on the same side of the line AB.

Dem.—Join CA, DA, CB, DB. Because A is the centre of the $\odot$ ECD, AC = AD; and because B is the centre of the

⊙ FCD, BC = BD; but this is contrary to Prop. VII. Hence



the ⊙s cannot intersect in more than one point on the same side of the line AB.

PROPOSITION IX.

3. **Dem.**—Because AD = AE, the ∠ AED = ADE; and because FE = FD, the ∠ FDE = FED. Now we have two △s ADF, AEF, having two sides AD, DF, and the contained ∠ ADF respectively = to the two sides AE, EF, and the contained ∠ AEF; ∴ (IV.) the ∠ DAF = EAF.

4. **Dem.**—Let G be the point where AF meets DE. Because AD = AE, and AG common, and the ∠ DAG = EAG; ∴ the ∠ AGD = AGE. Hence (Def. XIV.) AF is ⊥ to DE.

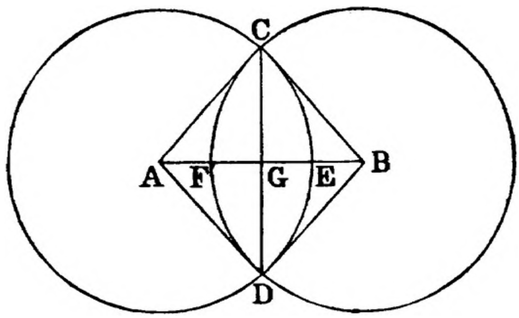
5. See Ex. 3, Prop. IV.

6. **Dem.**—Take any point G in AF, and from G let fall the ⊥ GH on AB. From AC cut off AJ = AH, and join GJ. Because AH = AJ, and AG common, and the ∠ HAG = JAG; ∴ (IV.) the ∠ AJG = AHG. Hence the ∠ AJG is right, and the base GH = GJ.

PROPOSITION X.

1. **Sol.**—Let AB be the given line. Take a part AE greater than half AB. With A as centre and AE as radius, describe the ⊙ CED. Take BF = AE. With B as centre and BF as radius, describe the ⊙ CFD, cutting the ⊙ CED in C, D. Join CD, cutting AB in G. AB is bisected in G.

Dem.—Join AC, BC, AD, BD. Because AC = BC, and CD common, and the base AD = BD; ∴ (VIII.) the ∠ ACD = BCD.



Again, since AC = BC, and CG common, and the ∠ ACG = BCG; ∴ (IV.) AG = BG.

2. **Dem.**—Take any point H equally distant from A, B. Join AH, BH, CH. Because AC = BC, and CH common, and the base AH = BH; ∴ (VIII.) the ∠ ACH = BCH. Hence any point equally distant from A, B, is in the bisector of the ∠ ACB.

PROPOSITION XI.

1. **Dem.**—Let the diagonals AD, BC, of the lozenge ABCD, intersect in E. Because AB = AC, and AD common, and the base BD = CD; ∴ (VIII.) the ∠ BAE = CAE. Again, AB = AC, AE common, and the ∠ BAE = CAE; ∴ (IV.) BE = CE, and the ∠ AEB = AEC. Hence AD bisects BC perpendicularly.

2. **Dem.**—Because DF = EF, the ∠ FED = FDE (V.), and CD = CE; ∴ (IV.) the △ DCF = ECF; ∴ the ∠ DCF = ECF, and (Def. XIV.) each of them is a right angle.

3. **Sol.**—Let AB be the given line. At the point A draw AC, making an angle with AB. In AC take AD = AB. At D erect DE ⊥ to AC. Bisect the ∠ BAC by AE, meeting DE in E. Join BE. BE is ⊥ to AB.

Dem.—AD = AB, AE common, and the ∠ DAE = BAE; ∴ (IV.) the ∠ ADE = ABE; but ADE is a right angle (const.); hence ABE is a right angle.

4. **Sol.**—Let AB be the given line, and C, D, the points. Join CD; bisect CD in E. Draw EF ⊥ to CD, meeting AB in F. F is the required point.

Dem.—Join CF, DF. Because (IV.) the △ CEF = DEF; ∴ FC = FD. Hence the point F is equally distant from C and D.

5. **Sol.**—Let AB be the given line, and C, D, the points. From C let fall a ⊥ CG on AB, and produce it to E, so that GE will be equal to CG. Join ED, and produce it to meet AB in F. F is the required point.

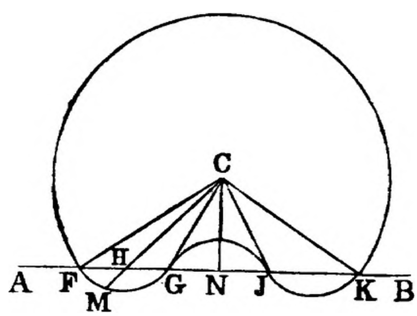
Dem.—Join CF. Because CG = EG, and GF common, and the ∠ CGF = EGF; ∴ (IV.) the ∠ CFG = EFG. Hence the ∠ CFD is bisected by the line AB.

6. **Sol.**—Let A, B, C, be the three given points. Join AB, BC. Bisect AB at D, and erect DF ⊥ to AB. Bisect BC at E, and erect EF ⊥ to BC. F is the required point.

Dem.—Join AF, BF, CF. Because AD = BD, and DF common, and the ∠ ADF = BDF; ∴ (IV.) AF = BF. In like manner BF = CF. Hence the three lines AF, BF, CF, are equal.

PROPOSITION XII.

1. **Dem.**—If possible let FGJK be a ⊙ meeting AB in the points F, G, J, K. Bisect FG in H. Join CH, and produce it to



M. Join CF, CG. Bisect GJ in N. Join CN, CJ, CK. Because FH = GH, and HC common, and the base FC = CG; ∴ the ∠ FHC = GHC, and (Def. XIV.) each of them is a right angle.

Again, since GN = JN, and CN common, and the base CG = CJ; ∴ the ∠ CNG = CNJ, and each is a right angle. Hence the ∠ CNH = CHN; ∴ CH = CN; but CN is greater than CK, because the point N is outside the ⊙; ∴ CH is greater than CK, and CM = CK; ∴ CH is greater than CM, which is absurd. Hence the ⊙ cannot meet AB in more than two points.

2. **Dem.**—Let ABC be the △, having the ∠ BAC equal to the sum of the ∠s ABC, ACB. Bisect AB in D, and erect DE ⊥ to AB, meeting BC in E. Join AE.

Because AD = BD, DE common, and the ∠ ADE = BDE; ∴ (IV.) the ∠ DAE = DBE; but the ∠ BAC = ABC + ACB; hence the ∠ EAC = ECA; ∴ each of the △s ABE, ACE, is isosceles; and since AE = BE = CE; ∴ BC = 2AE.

PROPOSITION XVII.

Dem.—Let ABC be the △. Take any point D in BC. Join AD. The ∠ ADC is greater than ABC (XVI.), and the ∠ ADB is greater than ACB; but ADC and ADB equal two right angles; ∴ ABC and ACB are less than two right angles.

PROPOSITION XVIII.

1. **Dem.**—Let ABC be the △, of which AC is greater than AB. From AC cut off AD = AB. With A as centre, and AB as radius, describe the circle ADE, cutting CD produced in E. Join AE. Now the ∠ ABC is greater than AEB; but AEB = ABE; ∴ ABC is greater than ABE, and ABE is greater than ACB (XVI.). Hence ABC is greater than ACB.

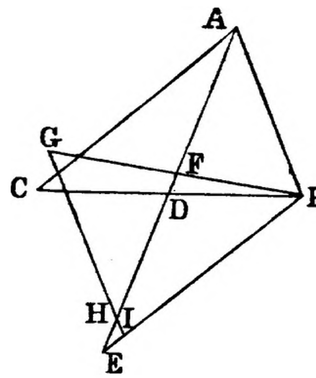
2. **Dem.**—Produce AB to D, so that AD = AC. Join CD. Now the ∠ ABC is greater than ADC (XVI.); but ADC = ACD; ∴ ABC is greater than ACD. Much more is ABC greater than ACB.

3. **Dem.**—Let ABCD be a quadrilateral, whose sides AB, CD, are the greatest and least. It is required to prove that the ∠ ADC is greater than ABC. Join BD. Because BC is greater than DC, the ∠ BDC is greater than DBC (XVIII.). Similarly the ∠ ADB is greater than ABD. Hence the ∠ ADC is greater than ABC.

4. **Dem.**—Let ABC be a △, whose side BC is not less than AB or AC. From A let fall a ⊥ AD on BC. Because BC is not less than AB, the ∠ BAC is not less than BCA; ∴ BCA must be acute. In like manner CBA must be acute. Hence AD must fall within the △ ABC.

PROPOSITION XIX.

1. **Dem.**—Bisect BC in D. Join AD; produce it to E, so that DE = AD. Join BE. Now the △ˢ BDE, ADC, have the sides BD, DE, of one respectively equal to CD, DA, of the other, and the contained ∠ˢ equal (xv.); ∴ (iv.) BE = AC, and the



∠ DBE = DCA; but the ∠ ABD is greater than DCA (hyp.); ∴ ABD is greater than EBD; hence the line BF which bisects the ∠ ABE falls above BC. Produce BF to G, and make GF = BF. Now, since ED = AD, EF is greater than AF. Cut off FH = AF. Join GH, and produce it to meet BE in I. Now we have in the △ˢ AFB, GFH, two sides AF, FB, in one equal HF, FG, in the other, and the contained ∠ˢ equal; hence AB = GH, and the ∠ ABF = HGF; but ABF = FBI (const.); ∴ BGI = GBI, and ∴ (v.) IB = IG; but EB is greater that IB, and IG greater than HG; ∴ EB is greater than GH, and we have proved BE = AC, and GH = AB. Hence AC is greater than AB.

2. **Dem.**—Take any point D in the base BC of an isosceles △ ABC. Join AD. Now the ∠ ADC is greater than ABD (xvi.), and ∴ greater than ACD. Hence (xix.) AC is greater than AD.

If we take the point D in the base produced, we have the ∠ ACB, that is, ABC greater than ADC; ∴ AD is greater than AB.

3. **Dem.**—This follows from the last exercise. For when we took the point in the base, and joined it to the vertex, the joining line was less than either side of the triangle; and when the point was in the base produced, the joining line was greater.

4. (1) **Dem.**—Let A be the given point, and EF the given line. From A let fall a ⊥ AB, and draw any other line AC to EF. The ∠ ACB is less than ABC (XVII); ∴ (XIX.) AC is greater than AB.

(2) **Dem.**—Take another point D in EF. Join AD. Now the ∠ ACD is greater than ABC, and therefore obtuse; hence ADC must be acute; ∴ AD is greater than AC.

5. **Dem.**—Because AB is greater than AC, the ∠ ACB is greater than ABC (XVIII.). Much more is the ∠ BCF greater than CBF. Hence (XIX.) BF is greater than CF. Again (hyp.), AB is greater than BC; but AB = CF (IV.); ∴ CF is greater than BC; ∴ (XVIII.) the ∠ CBF is greater than CFB, that is, than ABE. Hence ABE or CFB is less than half ABC.

PROPOSITION XX.

1. **Dem.**—Let ABC be a △. It is required to prove that the difference between two sides AB, AC, is less than BC. From AC cut off AD = AB, and join BD. Now AB and BC are greater than AD and DC; but AB = AD; ∴ BC is greater than DC, that is, greater than the difference between AB and AC.

2. **Dem.**—Let D be any point within a △ ABC. Join AD, BD, CD. Now (XX.) $DA + DB > AB$; $DB + DC > BC$; $DC + DA > AC$. Adding, we get $2\,(DA + DB + DC) > (AB + BC + CA)$; $\therefore (DA + DB + DC) > \left(\dfrac{AB + BC + CA}{2}\right)$.

3. **Dem.**—Let AD be the bisector of the ∠ BAC. Take any point E in AD. Join BE, CE. From AB cut off AF = AC, and join EF. Because AF = AC, and AE common, and the ∠ EAF = EAC; ∴ (IV.) the base EF = EC. Again, since EF = EC, the difference between BE and EC is equal to the difference between BE and EF; but BE − EF is less than BF (Ex. 1); ∴ BE − EC is less than BF; but BF is the difference between BA and AC. Hence the difference between BE and EC is less than the difference between BA and AC.

4. **Dem.**—Produce BA to F, so that AF = AC. Take any point E in the external bisector AD. Join EB, EC, EF. Now (IV.) EF = EC. To each add EB, and we have EF and EB = EC and EB; but EF and EB are greater than FB, that is, greater than AB and AC. Hence EB and EC are greater than AB and AC.

5. **Dem.**—Let ABCD be the polygon. Join BD. Now (xx.) AB + AD > BD; and BC + BD > CD; ∴ hence AB + AD + BC > CD.

6. **Dem.**—Let the Δ DEF be inscribed in ABC. Now (xx.) AD + AE > DE; EC + CF > EF; FB + BD > FD. Adding, we get (AB + BC + CA) > (DE + EF + FD).

7. **Dem.**—Let the polygon FGHJK be inscribed in the polygon ABCDE. Now (xx.) AF + AG > FG; BG + BH > GH; CH + CJ > HJ; DJ + DK > JK; EK + EF > KF. Adding, we get the perimeter of ABCDE greater than that of FGHJK.

8. **Dem.**—Let ABCD be a quadrilateral, AC, BD, its diagonals. Now, if AC, BD, are not equal, one of them must be the greater. Let BD be the greater; then we have the sum of the sides AB, BC, CD, DA, greater than 2BD, and ∴ greater than AC and BD.

9. **Dem.**—Let ABC be the Δ, AD one of its medians. Produce AD to E, so that ED = AD. Join EC. Now (IV.) EC = AB, and (xx.) AC and CE, that is, AC and AB, are greater than AE, that is, greater than 2AD. Similarly BC and CA are greater than 2CG, and AB and BC are greater than 2BF; ∴ (AB + BC + CA) > (AD + BF + CG).

10. **Dem.**—Let the diagonals AC, BD, of the quadrilateral ABCD intersect in E. Take any other point F in the quadrilateral. Join AF, BF, CF, DF. Now (xx.) BF + FD > BD, and AF + FC > AC. Adding, we get (AF + BF + CF + DF) > (AC + BD).

PROPOSITION XXI.

1. **Dem.**—Let ABC be the Δ, and O any point within it. Join OA, OB, OC. Now, AB + AC > OB + OC (XXI.); AC + BC > OA + OB; and AB + BC > OA + OC. Adding, we get 2(AB + BC + CA) > 2(OA + OB + OC); ∴ (OA + OB + OC) > (AB + BC + CA).

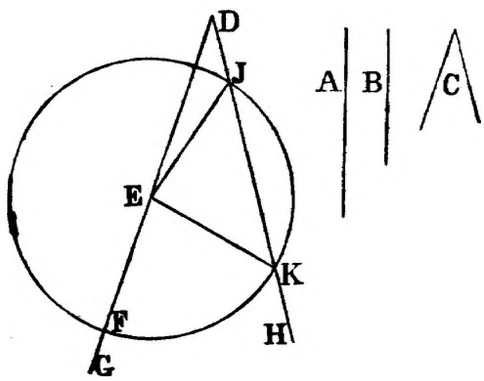
2. **Dem.**—Produce BC both ways to meet AM, DN, in E, F. Now (xx.) AE + EB > AB, and DF + FC > DC. To each add BC, and we have AE + EF + FD > AB + BC + CD. Again, EM + MN + NF > EF (Ex. 5, xx.). To each add AE and DF, and we get AM + MN + ND > AE + EF + FD; but we have shown that AE + EF + FD > AB + BC + CD; ∴ AM + MN + ND > AB + BC + CD.

PROPOSITION XXIII.

1. **Sol.**—Let A, B, be the given sides, and C the ∠ between them. Draw any line DG, and from DG cut off DE = A. At the point D in DG draw DH, making the ∠ GDH = C (XXIII.). In DH take DF = B, and join EF. DEF is the Δ required.

2. **Sol.**—Let AB be the given side, and D, E, the given angles. At the point A in AB make the ∠ BAC = D, and at the point B in AB make the ∠ ABC = E. ABC is the Δ required.

3. **Sol.**—Let A, B, be the given sides, and C the given angle, Draw any line DG, and in it make DE = A, and EF = B. At the point D in DG make the ∠ GDH = C. With E as centre,



and EF as radius, describe a ⊙, cutting DH in J, K. Join EK, EJ. Then evidently either of the Δˢ DEJ, DEK, will fulfil the given conditions.

4. (1) **Sol.**—Let AB be the base, C the given ∠, and S the sum of the sides. At the point A in AB make the ∠ BAF = C, and in AF take AE = S. Join BE. At the point B in BE make the ∠ EBG = BEG. ABG is the Δ required.

Dem.—Because the ∠ EBG = BEG; ∴ (VI.) EG = BG. To each add AG, and we have AG + GB = AE; but AE = S (const.); ∴ AG + GB = S.

(2) **Sol.**—Let AB be the base, C the given ∠, and D the difference of the sides. At the point A in AB make the ∠ BAG = C, and let AG = D. Produce AG to E. Join BG, and at the point B in BG make the ∠ GBE = EGB. AEB is the Δ required.

Dem.—Because the ∠ GBE = EGB; ∴ (VI.) EG = EB; but AE − GE = AG; ∴ AE − BE = AG = D. Hence the difference between AE and BE is D.

5. (1) Let A, B, be two points, one of which, B, is in the given line GF. It is required to find another point C in GF, such that CB + CA may be equal to a given line D.

Sol.—In GF take a part BE = D. Join AE, and at the point A in AE make the $\angle$ CAE = CEA; then C is the required point.

Dem.—Because the $\angle$ CAE = CEA, CA = CE (VI.). To each add CB, then CA + CB = BE; but BE = D; $\therefore$ CA + CB = D. Hence C is the required point.

(2) Let A, B, be the points, GF the given line.

Sol.—In GF take a part BG = D. Join AG, and at the point A in AG make the $\angle$ GAE = AGE. E is the required point.

Dem.—Because the $\angle$ GAE = AGE, GE = AE; $\therefore$ AE − EB = GE − EB; but GE − EB = GB; that is, equal to D. Hence AE − EB = D.

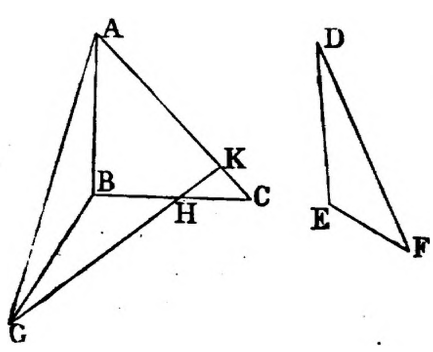
PROPOSITION XXIV.

1. **Dem.**—At the point A, in AB, make the $\angle$ BAH = EDF, and make AH = AC or DF. Join BH. Now (IV.) BH = EF. And because the $\angle$ BAC is greater than EDF, the bisector of the $\angle$ HAC must fall to the right of AB. Let AG be the bisector. Join HG. Now since AH = AC, and AG common, and the $\angle$ HAG = CAG; $\therefore$ (IV.) GH = GC. To each add BG, and we have BC = HG + GB; $\therefore$ (XX.) BC is greater than BH; that is, greater than EF.

2. **Dem.** (Diagram to Ex. 1).—The $\angle$ AHG = ACG; but AHG is greater than AHB; $\therefore$ ACG is greater than AHB; that is, greater than EFD.

PROPOSITION XXV.

1. **Dem.**—From BC cut off BH = EF. On BH describe the



$\triangle$ BGH = DEF; that is, having BG = DE, and GH = DF. Join

AG. Because BA = DE, and BG = DE; ∴ BA = BG; ∴ (VI.) the ∠ BGA = BAG. Produce GH to meet AC in K. Now since AC = DF, and GH = DF; ∴ AC = GH; ∴ GK is > AK; ∴ (XVIII.) the ∠ GAK is > AGK; but BAG = BGA; ∴ BAC is > BGH, that is, > EDF.

PROPOSITION XXVI.

1. Let ABC be the triangle.

Dem.—Let fall the ⊥ AD on BC. Now (XXVI.) the △ˢ ADB, ADC, are equal; ∴ DB = DC. Take any point E in AD. Join BE, CE. Now (IV.) the △ˢ BDE, CDE, are equal; ∴ BE = CE. Hence the point E is equally distant from the points A, B.

2. Let AD bisect the vertical ∠ BAC, and also the base BC.

Dem.—Produce AD to E, so that DE = AD. Join EC. Now (IV.) the △ˢ ADB, EDC, are equal; ∴ AB = CE, and the ∠ BAD = CED; but BAD = CAD (hyp.); ∴ CAD = CED; hence (VI.) CE = CA; but CE = BA; ∴ CA = BA. Hence the △ BAC is isosceles.

3. Let AB, AC, be two fixed lines, and D a point equally distant from them.

Dem.—Let fall ⊥ˢ DE, DF, on AB, AC. Join EF, AD. Because DE = DF, the ∠ DFE = DEF; but the ∠ DFA = DEA; ∴ the ∠ AFE = AEF, and ∴ AE = AF. Now AE = AF, AD common, and the base DE = DF; ∴ the ∠ EAD = FAD; ∴ the bisector of the ∠ BAC is the locus of the point D. In like manner, if we produce BA to G, the locus of a point equally distant from AC, AG, will be the bisector of the ∠ CAG.

4. Let AB be the given right line, and CD, EF, the other lines.

Sol.—Let CD, EF, intersect in G, and meet AB in H, J. Bisect the ∠ HGJ by GK, meeting AB in K. K is the point required.

Dem.—Let fall ⊥ˢ KM, KN, on CD, EF. Because the ∠ NGK = MGK, and GNK = GMK, and GK common; ∴ (XXVI.) KN = KM.

5. Let ABC, DEF, be two right-angled △ˢ, having the base BC = EF, and the acute ∠ ABC = DEF.

Dem.—The △ˢ ABC, DEF, have the ∠ˢ BAC, ABC, equal to the ∠ˢ EDF, DEF, and the side BC = EF; ∴ (XXVI.) they are equal in every respect.

6. Let the right-angled $\triangle^s$ ABC, DEF, have the sides AB, DE, equal, and also their hypotenuses BC, EF equal. It is required to prove that the $\triangle^s$ are equal in every respect.

Dem.—At the point B in BC make the $\angle$ GBC = DEF (xxiii.), and make BG = DE or AB. Join CG, AG.

Now the $\triangle^s$ GBC, DEF, have the sides GB, BC = DE, EF, and the $\angle$ GBC = DEF; ∴ (iv.) CG = DF, and the $\angle$ BGC = EDF; but EDF is a right $\angle$; ∴ BGC is right, and ∴ = BAC. Now BG = DE, and DE = AB; ∴ BG = AB; ∴ the $\angle$ BAG = BGA; but BAC = BGC; ∴ CAG = CGA; hence CG = CA; but CG = DF; ∴ AC = DF. Hence the $\triangle^s$ ABC, DEF, are equal in every respect.

7. Let ABC be the $\triangle$, and let the bisectors of the $\triangle^s$ ABC, ACB, meet in O. Join OA. It is required to prove that OA bisects the $\angle$ BAC.

Dem.—From O let fall $\perp^s$ OD, OE, OF, on AB, BC, CA. Join DF. The $\triangle^s$ OBD, OBE, are equal (xxvi.); ∴ OD = OE. Similarly OE = OF; ∴ OF = OD, and ∴ (v.) the $\angle$ ODF = OFD; but the $\angle$ ODA = OFA (const.); ∴ the $\angle$ ADF = AFD; ∴ (vi.) AF = AD. Now AF = AD, AO common, and the base OF = OD; hence (viii.) the $\angle$ OAF = OAD. Therefore AO is the bisector of the $\angle$ BAC.

8. Let ABC be the $\triangle$, and let BO, CO, bisecting the two external $\angle^s$ meet in O. Join OA. It is required to prove that OA bisects the $\angle$ BAC.

Dem.—From O let fall $\perp^s$ OD, OE, OF, on AB, BC, CA. Join DF. Now, as in the last Exercise, OD = OF; ∴ the $\angle$ OFD = ODF; but the $\angle$ OFA = ODA; ∴ AFD = ADF, and ∴ AD = AF. Now AD = AF, AO common, and the base OD = OF; ∴ the $\angle$ OAD = OAF. Therefore AO bisects the $\angle$ BAC.

9. Let A, B, C, be the given points. It is required to draw a line through C, such that the $\perp^s$ on it from A, B, may be equal.

Sol.—Join AB; bisect it in O. Join CO, and produce it to D. From A, B, let fall the $\perp^s$ AE, BF, on CD.

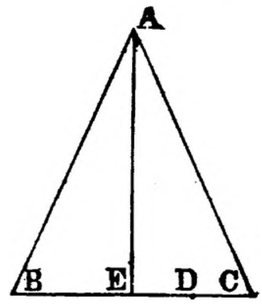
Dem.—Because AO = BO, and the $\angle^s$ AEO, AOE = BFO, BOF; ∴ (xxvi.) AE = BF.

10. Let AB, AC, be the given lines, and D the given point.

Sol.—Bisect the $\angle$ BAC by AE. From D let fall a $\perp$ DE on AE, and produce it both ways to meet AB, AC, in B, C.

Dem.—The $\triangle^s$ ABE, ACE, have the $\angle^s$ AEB, EAB, equal to the $\angle^s$ AEC, EAC, and the side AE common; ∴ the $\angle$ ABE

= ACE. Hence the △ ABC is isosceles. There are two solutions. For if we produce BA to F, bisect the ∠ CAF by AG,



and from D let fall the ⊥ DH on AG, and produce it to meet BA in F, we will have another isosceles triangle.

PROPOSITION XXIX.

1. (1) **Dem.**—If AB, CD, are not parallel, let them meet in K. Then we have the exterior ∠ EGK of the △ GKH equal to the interior ∠ GHK; but this is impossible (XVI.). Therefore AB, CD, must be parallel.

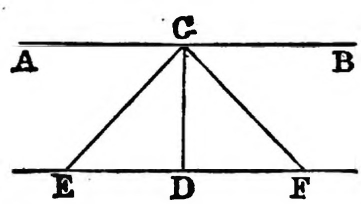
(2) If AB, CD, are not parallel, let them meet in K. Then we have the ∠ˢ KGH, GHK, of the △ GHK, equal to two right angles, which is impossible (XVII.). Hence AB, CD, must be parallel.

2. Let AB, CD, be the ∥ lines, and AC, BD, the ⊥ˢ intercepted between them.

Dem.—Join AD. Now, the ∠ ACD is right (hyp.), and ABD, CDB, together equal two right ∠ˢ (XXIX.); but CDB is right; ∴ ABD is right, and hence = ACD, and the ∠ BAD = ADC (XXIX.). Therefore the △ˢ ABD, ACD, have two ∠ˢ of one equal to two ∠ˢ of the other, and the side AD common. Hence (XXVI.) BD = AC.

3. Let EF be ∥ to AB.

Dem. Bisect the ∠ ACD, BCD, by CE, CF. Now (XXIX.)



the ∠ ACE = DEC; but ACE = DCE; ∴ DEC = DCE, and ∴ DC = DE. In like manner DC = DF. Therefore DE = DF.

4. Let EF be the line whose middle point is O, and terminated by the parallels AB, CD.

Dem.—Through O draw a line GH, meeting AB, CD in G, H. The ∠ GOE = HOF (xv.), and the ∠ GEO = OFH (xxix.), and OE = OF (hyp.); therefore (xxvi.) OG = OH.

5. Let AB, CD, be the ∥s, and O the point equidistant from them.

Dem.—Through O draw EF, meeting AB, CD, in E, F, and draw GH, JK, meeting them in G, H, J, K. Because EF is bisected in O, ∴ (4) GH, JK, are bisected in O; then the △s GOJ, HOK, have two sides GO, OJ, and the ∠ GOJ in one equal to the sides HO, OK, and the ∠ HOK in the other. Hence (iv.) GJ = HK.

6. Let AEFD be the parallelogram formed by drawing parallel lines from a point F in BC to the sides AB, AC, of the equilateral △ ABC.

Dem.—The ∠ EFB = ACB (xxix.); ∴ EFB is an ∠ of an equilateral △, and EBF is an ∠ of an equilateral △ (hyp.); ∴ EBF is an equilateral △; ∴ EF = BF; but EF = AD; ∴ EF + AD = 2BF. In like manner, AE + DF = 2CF. Hence AE + AD + FE + FD = 2BC.

7. Let ABCDEF be the hexagon, and let its diagonals AD, BE intersect in O. Join CO, FO. It is required to prove that CO, FO are in one straight line.

Dem.—The ∠ ABO = DEO (xxix.), and the ∠ AOB = DOE (xv.), and the side AB = DE (hyp.); ∴ (xxvi.) BO = EO. Again (xxix.) the ∠ CBO = FEO, and CB = EF (hyp.), and we have shown that BO = EO; ∴ (iv.) the ∠ BOC = EOF; to each add the ∠ FOB, and we have BOC + FOB = EOF + FOB; but EOF + FOB = two right angles (xiii.); ∴ BOC + FOB = two right angles, and ∴ (xiv.) CO, OF are in one straight line.

PROPOSITION XXXI.

1. Let A, B, be the given ∠s, and H the altitude.

Sol.—Draw any line CD, and make the ∠ DCE = A, and the ∠ CDE = B; let fall a ⊥ EF on CD. If EF = H, the △ is constructed. If not, produce it, and cut off EG = H. Through G draw JK ∥ to CD, and produce EC, ED, to meet it in J, K.

Dem.—The $\angle$ EJK = ECD (xxix.) = A. In like manner EKJ = B, and EG = H. Therefore EJK is the $\triangle$ required.

2. Let AB be the given line, C the given point, and M the given $\angle$.

Sol.—Through C draw CE $\parallel$ to AB (xxx.). At the point C in CE make the $\angle$ ECD = M. The $\angle$ ECD = CDA (xxix.) $\therefore$ CDA = M.

3. **Dem.**—The $\angle$ CAD = ADE (xxix.); but CAD = EAD (const.); $\therefore$ ADE = EAD, and $\therefore$ EA = ED. In like manner FB = FD. Again, the $\angle$ CAB = DEF (xxix.); but CAB is an $\angle$ of an equilateral $\triangle$; $\therefore$ DEF is an $\angle$ of an equilateral $\triangle$. Similarly DFE is an $\angle$ of an equilateral $\triangle$; hence DEF is an equilateral $\triangle$; $\therefore$ DE = EF; but DE = AE; $\therefore$ AE = EF. In like manner BF = EF. Hence AB is trisected.

4. Let ABC be the equilateral triangle.

Sol.—Let fall a $\perp$ AD on BC. Bisect the $\angle$ BAD by AE, meeting BC in E. Through E draw EF $\parallel$ to AD, meeting AB in F. Through F draw FG $\parallel$ to BC, and complete the $\square$ EFGH. EFGH is a square.

Dem.—The $\angle$ FEA = EAD (xxix.), = FAE; $\therefore$ FA = FE; but FAG is an equilateral $\triangle$, because FG is $\parallel$ to BC; $\therefore$ AF = FG; but AF = EF; $\therefore$ EF = GF, and EF = GH, and GF = EH; $\therefore$ the four sides are equal, and (xxix.) the $\angle$ GFE = BEF; but BEF is a right $\angle$, $\therefore$ GFE is right. Hence EFGH is a square.

5. (1) Let ABC be the triangle.

Sol.—Produce AB to G. Bisect the $\angle$ GBC by BF, meeting AC produced in F. Through F draw FG $\parallel$ to BC.

Dem.—The $\angle$ CBF = BFG (xxix); but CBF = GBF (const.); $\therefore$ GBF = BFG, and $\therefore$ FG = BG. If we bisect the $\angle$ BCF, we get another solution.

(2) **Sol.**—Produce AB, AC to E, F. Bisect the $\angle$s CBE, BCF; and through D, where the bisectors meet, draw EF $\parallel$ to BC, meeting AE, AF, in E, F.

Dem.—The $\angle$ CBD = EDB (xxix.); but CBD = EBD (const.); $\therefore$ EDB = EBD; and $\therefore$ (vi.) EB = ED. Similarly, FC = FD. Hence EB + FC = EF.

If we bisect the $\angle$s ABC, ACB, we have another solution.

(3) **Sol.**—Produce the base BC to G. Bisect the $\angle$s ABC, ACG, by BD, CD. Through D draw DF $\parallel$ to BC, meeting AB, AC in F, E.

Dem.—The ∠ FDB = CBD (xxix.); but CBD = FBD (const.); ∴ FBD = FDB; and therefore FB = FD. In like manner CE = DE. Hence BF − CE = FD − DE = FE. If we produce CB to H, and bisect the ∠s ACB, ABH, we will have another solution.

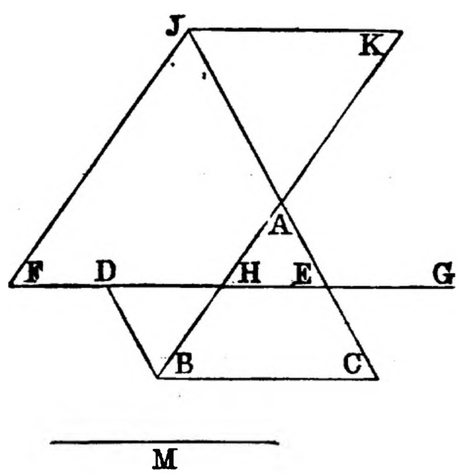
6. Let AB, DC be the ∥ lines, and B, D the given points.

Sol.—Join BD; bisect it in E. At E erect EA ⊥ to BD; produce it to meet CD in C. Join AD, BC.

Dem.—Because EB = ED, EA common, and the ∠ AEB = AED; ∴ (IV.) AB = AD. In like manner BC = DC, and the four sides are equal to each other. Hence (Def. xxix.) ABCD is a lozenge.

7. Let AB, AC be the lines given in position, M the line of given length, and FG the line to which the required line is to be parallel.

Sol.—(1) In FG take a part DE = M; through D draw DB ∥ to AC, and through B draw BC ∥ to DE. BC fulfils the required conditions.



Dem.—Because DBCE is a parallelogram, BC = DE; but DE = M; ∴ BC = M.

(2) **Sol.**—In FG take FH = M. Through F draw FJ ∥ to AB, meeting CA produced in J; and through J draw JK ∥ to FH. JK fulfils the required conditions.

Dem.—Because FJKH is a parallelogram, FH = JK; but FH = M; ∴ JK = M.

PROPOSITION XXXII.

1. Let ABC be the right angle.

Sol.—Make the ∠ ABD equal an ∠ of an equilateral △ (XXIII.), and draw BE bisecting it.

Dem.—Because the ∠ ABD is an angle of an equilateral △, it is two-thirds of a right ∠; ∴ CDB is one-third, and half ABD is one-third. Hence ABC is trisected.

2. (1) Let ABC be the triangle.

Dem.—Draw the median AD. Now if BD be greater than AD, the ∠ BAD will be greater than ABD (XVIII.) Similarly the ∠ CAD will be greater than ACD. Hence the ∠ BAC will be greater than ABC + BCA, and ∴ will be obtuse, when the side BC is greater than 2AD.

(2) **Dem.**—If BD = AD, the ∠ BAD = ABD; and if CB = AD, the ∠ CAD = ACD. Hence the ∠ BAC is = ABC + BCA, and ∴ right when BC = 2AD.

(3) In like manner it can be shown that the ∠ BAC is acute, when BC is less than 2AD.

3. Let ABCDE be the polygon.

Dem.—Produce AB, DC to meet in A′; BC, ED to meet in B′, &c.

Now the sum of the ∠s of the △ BA′C is two right ∠s; similarly the sum of the ∠s of each of the external △s is two right ∠s. Hence if there be n external triangles, the sum of their ∠s will be $2n$ right ∠s; but the sum of the exterior ∠s BCA′, CDB′, &c., is four right ∠s; and the sum of the exterior ∠s CBA′, DCB′, &c., is four right ∠s. Hence the sum of the remaining ∠s must be $(2n - 8)$ right ∠s; that is, $2(n-4)$ right ∠s.

4. Let BAC be the triangle.

Dem.—Produce BA to D, and bisect the ∠ CAD by the line AE ∥ to BC.

The ∠ EAC = ACB (XXIX.); but EAC = EAD, and EAD = ABC; ∴ ACB = ABC. And hence AB = AC.

5. Let E be the point where CD cuts AB.

Dem.—Bisect AB in F. Join CF, DF. Now the lines AF, BF, CF, DF are equal (XII., Ex. 2). And because FD = FB, the ∠ FBD = FDB = FDE + EDB; to each add the ∠ EDB; then the ∠s EBD + EDB = FDE + 2EDB; but the ∠ CEB = EBD + EDB (XXXII.); ∴ CEB = FDE + 2EDB; but CEB = FCB + CFE, and FCD = FDE; ∴ CFE = 2EDB. Again,

CFE = ACF + CAF; but ACF = CAF (v.); ∴ CFE = 2CAF, ∴ 2CAF = 2EDB. And hence CAF = EDB.

6. Let ABC be the triangle.

From B, C draw ⊥ˢ BD, CE to the sides AC, AB, and let them meet in G; join AG, and produce it to meet BC in F. It is required to prove that AF is ⊥ to BC.

Dem.—Join DE. Now we have two right-angled triangles BEC, BDC, and we have joined their vertices E, D; hence (5) the ∠ EDB = ECB. Similarly from the △ˢ AEG, ADG, the ∠ EAG = EDG (5); ∴ EAG = FCG, and AGE = CGF (xv.); hence (Cor. 2) the ∠ AEG = GFC; but AEG is a right angle; ∴ CFG is right; and hence AF is ⊥ to BC.

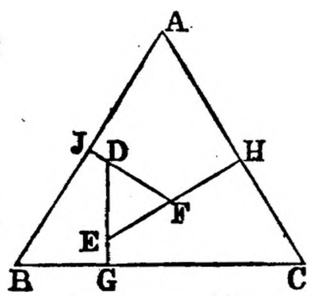
7. Let ABCD be the ▭, and BE, CE the bisectors of the adjacent angles B, C. It is required to prove that the ∠ BEC is right.

Dem.—The ∠ˢ ABC, DCB equal two right angles (xxix); ∴ EBC + ECB equal a right angle; and hence the ∠ BEC is right.

8. Let ABCD be the quadrilateral. Bisect the external ∠ˢ A, B, C, D; let the bisectors meet in E, F, G, H. It is required to prove that the ∠ˢ EHG, EFG, of the quadrilateral EFGH, are together equal to two right angles.

Dem.—Produce BA, CD to J, K. Now the ∠ˢ ADC, ADK, DAB, DAJ equal four right angles; and the ∠ˢ DHA, HAD, ADH equal two right angles; ∴ the ∠ˢ of the △ HAD equal half sum of the ∠ˢ ADC, ADK, DAB, DAJ; but the ∠ˢ HAD, ADH are the halves of JAD, ADK; hence the ∠ DHA is half sum of BAD, ADC; in like manner BFC is half sum of ABC, BCD. Hence the sum of the angles DHA, BFC is half sum of the four angles of the quadrilateral ABCD, and ∴ equal to two right angles.

9. Let the sides of the triangle DEF be perpendicular to the



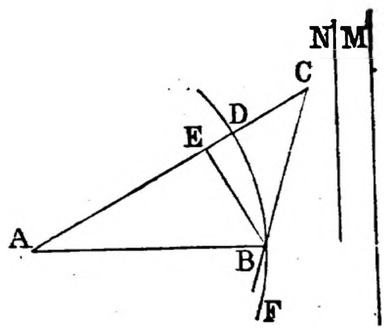
sides of ABC. It is required to prove that the △ˢ DEF, ABC are equiangular.

Dem.—Since the $\angle$s CHE, EGC are right, the sum of the $\angle$s HCG + HEG = two right $\angle$s (Cor. 3), and HED + HEG = two right $\angle$s. Reject the common $\angle$ HEG, and we have the $\angle$ HCG = DEF, that is, the $\angle$ ACB = DEF. In like manner the $\angle$ BAC = EFD, and ABC = EDF.

10. (1) Let M equal sum of sides, and N the hypotenuse.

Sol.—Draw any line AC, and make it equal to M. In AC take a part AD = N. At the point C in AC make the $\angle$ ACB equal half a right angle. With A as centre, and AD as radius, describe the $\odot$ DBF, cutting CB in B. Join AB, and at the point B in BC make the $\angle$ EBC = ACB. AEB is the required triangle.

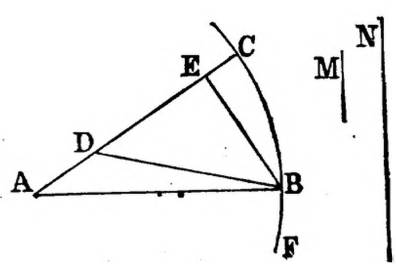
Dem.—Because the $\angle$ EBC = ACB, EC = EB (VI.). To each add AE, and we have AC = AE + EB; but AC = M (const.);



$\therefore$ AE + EB = M. Again, the $\angle$ AEB = EBC + ECB (XXXII.); but EBC = ECB; $\therefore$ AEB = 2ECB, and is therefore a right angle.

(2) Let M equal differences of sides, and N the hypotenuse.

Sol.—Draw any line AC = N. In AC take AD = M. At the point D in AC make the $\angle$ CDB = half a right angle. With A as centre, and AC as radius, describe the $\odot$ CBF, cutting DB in B. From B let fall the $\perp$ BE on AC. Join AB. AEB is the required triangle.



Dem.—Because the $\angle$ AEB is right, and EDB half right, $\therefore$ EBD is half right, and (VI.) ED = EB. Hence AD is the

difference between AE and EB. Again, AC = AB; but AC = N. Hence AB = N.

11. Let ABC be an isosceles $\triangle$. From B let fall a $\perp$ BD on AC. It is required to prove that the $\angle$ DBC = half BAC.

Dem.—Bisect the $\angle$ BAC by AE, meeting BC in E, and BD in F. Now the $\angle$ BFE = AFD (xv.), and BEF = ADF (ix., Ex. 2); hence (Cor. 2) EBF = FAD; but FAD = half BAC. Therefore EBF = half BAC.

12. Let ABC be a triangle. Produce BC to E. Bisect the $\angle$s ABC, ACE by BD, CD. It is required to prove that the $\angle$ BDC = half BAC.

Dem.—The $\angle$ ACE = ABC + BAC (xxxii.), and DCE = DBC + BDC; but DCE = $\frac{1}{2}$ ACE; $\therefore$ DBC + BDC = $\frac{1}{2}$ (ABC + BAC); but DBC = $\frac{1}{2}$ ABC. Hence BDC = $\frac{1}{2}$ BAC.

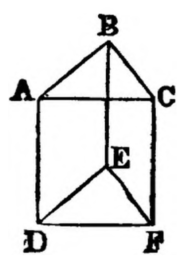
13. **Dem.**—Because AB = AD, the $\angle$ ADB = ABD; but ADB = ACB + CBD (xxxii.); $\therefore$ ABD = ACB + CBD. To each add the $\angle$ CBD, and we have the $\angle$ ABC = ACB + 2CBD; $\therefore$ 2 CBD = ABC − ACB; and hence CBD = $\frac{1}{2}$ (ABC − ACB).

14. **Dem.**—Produce BA, BC to D, E. Bisect the $\angle$s CAD, ACE by AF, CF.

Now the $\angle$ ACE = (A + B) (xxxii.); $\therefore$ ACF = $\frac{1}{2}$ (A + B). Similarly the $\angle$ CAF = $\frac{1}{2}$ (B + C). Hence the $\angle$ AFC = $\frac{1}{2}$ (C + A).

PROPOSITION XXXIII.

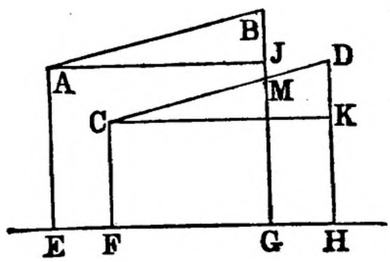
1. **Dem.**—Join AD, BE, CF. Now, because AB is equal and parallel to DE, $\therefore$ (xxxiii.) AD is equal and parallel to BE.



In like manner CF is equal and parallel to BE; hence CF is equal and parallel to AD; and $\therefore$ (xxxiii.) AC is equal and parallel to DF.

2. (1) Let AB, CD be equal and parallel lines, and EH any other line. From A, B, C, D let fall $\perp$s AE, BG, CF, DH on EH. It is required to prove that EG = FH.

Dem.—Through A, C draw AJ, CK ∥ to EF.



Now, because AJ, CK are each ∥ to EF, they are ∥ to one another, and AB is ∥ to CD; hence (XXIX., Ex. 8) the ∠ BAJ = DCK, also the ∠ AJB = CKD, because each is right, and the side AB = CD; ∴ (XXVI.) AJ = CK; but AJ = EG, and CK = FH. Hence EG = FH.

(2) As in (1) the ∠ˢ BAJ, AJB are respectively equal to the ∠ˢ DCK, CKD, and the side AJ = CK. Hence AB = CD.

3. **Dem.**—Since AB = CD, and AJ = CK, and the ∠ AJB = CKD, each being right; ∴ (XXVI., Ex. 6) the △ˢ ABJ, CDK are equal in every respect; hence the ∠ ABJ = CDK; but CDK = CMG (XXIX.); ∴ ABJ = CMG. Hence AB is parallel to CD.

4. Let AB, CD be the equal and parallel lines. Join AD, BC, intersecting in E. It is required to prove that AD, BC bisect each other in E.

Dem.—The ∠ˢ ABE, BAE are respectivly equal to the ∠ˢ DCE, EDC, and the side AB = CD (hyp.). Hence (XXVI.) BE = CE, and AE = DE.

PROPOSITION XXXIV.

1. See last exercise to Prop. XXXIII.

2. Let ABCD be the ▭; AC, BD its diagonals, which are equal. It is required to prove that the ∠ˢ of ABCD are right ∠ˢ.

Dem.—Because AD = BC, and AB common, and the bases BD, AC equal, ∴ (VIII.) the ∠ BAD = ABC; but (XXIX.) BAD + ABC equal two right angles; hence each is right, and (XXXIV.) the ∠ BAD = BCD, and ABC = ADC. Therefore all the ∠ˢ are right angles.

3. See "Sequel to Euclid," Prop. XV., p. 11, 3rd Edition.

4. Let AB, CD be two ∥ lines, of which AB is the greater. Join AC, BD. It is required to prove that AC, BD produced will meet.

Dem.—From BA cut off EB = CD. Join EC. Because EB is equal and parallel to CD; ∴ (XXXIII.) EC is equal and parallel to BD; and ∴ (XXIX.) the ∠ AEC = ABD. To each add the ∠ CAE; then CAE + AEC = CAE + ABD; but CAE and AEC are less than two right angles (XVII.); hence CAE and ABD are less than two right angles. And ∴ AC, BD, if produced, will meet.

5. Let ABCD be a quadrilateral, having AB, CD parallel, but not equal; and AC, BD equal, but not parallel. It is required to prove that the ∠s CAB, CBD are supplemental.

Dem.—In CD take CE = AB. Join BE. Now (XXXIII.) AC is = and ∥ to BE; but AC = BD (hyp.); ∴ BE = BD; and ∴ (V.) the ∠ BDE = BED, and (XXXIV.) the ∠ CAB = CEB; hence the ∠s CAB + BDE = CEB + BED. But CEB and BED are supplemental; hence CAB and BDE are supplemental.

6. Let A, B, C be the middle points of the sides.

Sol.—Join AB, BC, CA; and through the points A, B, C draw DE, EF, FD ∥ to BC, AC, AB. DEF is the required triangle.

Dem.—AB = CD (XXXIV.), and AB = CF; hence CD = CF. In like manner AD = AE, and BF = BE.

7. Let ABCD be a quadrilateral, whose diagonals are AC, BD. Through B, D, draw FG, EH ∥ to AC, and through C, A, draw GH, EF ∥ to BD. Join FH. It is required to prove that the area of the △ EFH is equal to the area of ABCD.

Dem.—The area of the △ EFH is half the area of the ▭ EFGH (XXXIV.), and the area of ABCD is half the area of EFGH; ∴ EFH = ABCD; and the sides EF, EH are equal to BD, AC; and the ∠ FEH = AJD, which is the ∠ between AC, BD.

PROPOSITION XXXVI.

Dem.—Produce AB, EF to meet in J. Through J draw JK ∥ to AH or BG, and produce DC to meet it in K. Join KG. Now JK = BC (XXXIV.); but BC = FG (hyp.); ∴ JK = FG, and it is ∥ to it; hence JFGK is a ▭; ∴ JF is ∥ to KG; but JE is ∥ to GH. Hence KG, GH are in one straight line; ∴ JEHK is a ▭ and it is equal to JADK (XXXV.); but JBCK = JFGK. Hence ABCD = EFGH.

PROPOSITION XXXVII.

1. See "Sequel to Euclid," Prop. VI., p. 4, 3rd Edition.

2. Let ABCD be a given quadrilateral. It is required to construct a triangle equal in area to ABCD.

Sol.—Join AC. Produce DC to E; and through B draw BE ∥ to AC. Join AE. ADE is the △ required.

Dem.—The △s ABC, AEC are equal (XXXVII.) To each add the △ ACD, and we have the △ ADE equal to the quadrilateral ABCD.

3. Let the pentagon ABCDE be the given rectilineal figure. It is required to construct a △ equal in area to ABCDE.

Sol.—Join AC, AD. Through B, E draw BF, EG ∥ to AC, AD, and meeting DC produced both ways in F, G. Join AF, AG. AGF is the △ required.

Dem.—The △s ABC, AFC are equal (XXXVII.); to each add ACDE, and we have the pentagon ABCDE equal to the quadrilateral AFDE. Again (XXXVII.), the △ AGD = AED. To each add the △ ADF, and we have the △ AGF equal to the quadrilateral AFDE; but AFDE = ABCDE. Hence AGF = ABCDE.

4. Let ABCD be a given parallelogram. It is required to construct a lozenge equal to ABCD, and having CD as base.

Sol.—If AD = DC, the thing required is done. If not, let DC be the greater. With D as centre, and DC as radius, describe a ⊙ ECG, cutting AB in E. Join DE. Through C draw CF ∥ to DE, meeting AB produced in F. DEFC is the required lozenge.

Dem.—DE = DC; but DC = EF (XXXIV.); ∴ DE = EF. Hence the four sides are equal; ∴ DEFC is a lozenge, and (XXXV.) is equal to ABCD.

5. Let ABC be a △, whose base BC is given, and whose area is given. It is required to find the locus of its vertex A.

Sol.—Through A draw DE ∥ to BC. DE is the required locus.

Dem.—Take any other point F in DE. Join BF, CF. Now (XXXVII.) the △s ABC, FBC are equal. Hence DE is the locus of the vertex of all triangles, having BC as base, and whose area is equal to the area of the △ ABC.

6. **Dem.**—Through E draw EG ∥ to FD, and meeting AD in G. Join GF, GC. Now (XXXVII.) the △ EFD = GFD; but GFD = GCD, and GCD is less than ACD; ∴ EFD is less than ACD, that is, is less than half ABCD.

PROPOSITION XXXVIII.

1. Let ABC be the △, and AD one of its medians. It is required to prove that AD bisects the triangle.

Dem.—BD = CD (Def. Prop. xx.); ∴ (xxxviii.) the △ ABD = ACD.

2. Let ABC, DEF be two △s, having the sides AB, BC equal o the sides DE, EF, and the contained ∠s supplemental. It is required to prove that the △s are equal.

Dem.—Produce CB to G, and make BG = BC or EF. Join AG. Now the ∠s ABC, DEF are supplements (hyp.), and ABC, ABG are supplements (xiii.) Reject ABC, and we have ABG = DEF; hence (iv.) the △ ABG = DEF; but ABG = ABC xxxviii.) Hence DEF = ABC.

3. **Dem.**—Divide the base BC of the △ ABC into any number, such as four equal parts, in the points D, E, F. Join AD, AE, AF. It is required to prove that the four △s into which ABC is divided are equal.

The △ BAD = EAD (xxxviii.) Similarly EAD = EAF, and EAF = CAF. Hence the four △s are equal.

4. Let ABDC be a ▭, whose diagonals AD, BC intersect in F. In BC take a point E. Join EA, ED. It is required to prove that the △ ABE = DBE, and that ACE = DCE.

Dem.—AF = DF (xxxiv., Ex. 1); hence (xxxviii.) the △ AFB = DFB, and AFE = DFE; hence AEB = DEB; but ABC = DBC; ∴ AEC = DEC.

5. Let ABCD be a quadrilateral; and let AC, one of its diagonals, bisect the other, BD in E. It is required to prove that AC bisects ABCD.

Dem.—The △ AEB = AED (xxxviii.), and the △ CEB = CED. Hence ABC = ADC.

6. See "Sequel to Euclid," Prop. xiii., p. 10, 3rd Edition.

7. See "Sequel to Euclid," Prop. xiii., p. 10, Cor. 1.

8. See "Sequel to Euclid," Prop iii., Cor 1, p. 2.

9. Let ABC be a △; D, E the middle points of AB, AC; F any point in BC. Join DE, EF, FD. It is required to prove that DEF = $\frac{1}{4}$ ABC.

Dem.—Bisect BC in G. Join DG, EG. Now (xxxvii.) the △ DEF = DEG; but DEG = $\frac{1}{4}$ ABC (8). Hence DEF = $\frac{1}{4}$ ABC.

10. Let ABC be a given △, and D a given point in BC. It is required to draw a line through D, bisecting the △ ABC.

Sol.—Join AD. Bisect BC in E. Through E draw EF ∥ to AD, and meeting AB in F. Join DF. DF is the required line.

Dem.—Join AE. Now (xxxvii.) the △ˢ EFD, EFA are equal. To each add the △ BEF, and we have the △ BFD = BAE; but BAE = $\frac{1}{2}$ BAC. Hence BFD = $\frac{1}{2}$ BAC.

11. Let ABC be a given △, and D a given point within it. It is required to trisect ABC by three lines drawn from D.

Sol.—Trisect BC in E, F (xxxiv., Ex. 3) Join AD, DE, DF. Through A draw AG, AH ∥ to DE, DF. Join DG, DH. AD, DG, DH trisect ABC.

Dem.—Join AE, AF. Now (xxxvii.) the △ˢ ADG, AEG are equal. To each add the △ AGB, and we have the quadrilateral ADGB equal to the △ AEB; but AEB = $\frac{1}{3}$ ABC (3); hence ADGB = $\frac{1}{3}$ ABC. In like manner ADHC = $\frac{1}{3}$ ABC; ∴ the △ DGH = $\frac{1}{3}$ ABC. Hence the △ ABC is trisected by the lines AD, GD, HD.

12. Let ABCD be a ▭, whose diagonals AC, BD intersect in E. Through E draw any line FG, meeting AB, CD in F, G. It is required to prove that FG bisects ABCD.

Dem.—The ∠ BEF = GED (xv.), and the ∠ FBE = GDE (xxix.), and the side EB = ED (xxxiv., Ex. 1); hence (xxvi.) the △ˢ BEF, DEG are equal. Similarly, AEF = CEG, and AED = CEB. Hence FG bisects ABCD.

13. Let ABCD be a trapezium. Bisect AD in E. Join EB, EC. It is required to prove that the △ BEC = $\frac{1}{2}$ ABCD.

Dem.—Produce BE, CD to meet in F. Now (xxvi.) the △ AEB = DEF, and EB = EF. And since AEB = DEF, AEB + CED = CEF; but (xxxviii.) CEF = BEC. Hence BEC = AEB + CED.

PROPOSITION XL.

1. Let ABC, DEF be two △ˢ whose bases and altitudes are equal. It is required to prove that the △ˢ are equal.

Dem.—Produce BC; and in BC produced cut off GH = EF or BC, and construct the △ JGH, having its sides JG, GH, HJ respectively equal to the sides DE, EF, FD of the △ DEF. Join AJ; and from A, J let fall ⊥ˢ AL, JK on BH. Because the △ DEF = JGH, their altitudes are equal; but the altitudes of DEF and ABC are equal (hyp.); hence the altitudes of JGH

and ABC are equal; that is, $JK = AL$, and they are parallel; hence (XXXIII.) AJ, BH are parallel; ∴ (XXXVIII.) the △ ABC = JGH; but JGH = DEF. Hence ABC = DEF.

3. See "Sequel to Euclid," Prop. II., p. 2.

4. See "Sequel to Euclid," Prop. III., Cor. 1, p. 2.

5. See "Sequel to Euclid," Prop. II., Cor., p. 2.

6. See "Sequel to Euclid," Prop. V., p. 3.

7. Let ABCD be a trapezium, whose opposite sides AD, BC are ∥; E, F the middle points of AB, DC. Join EF. It is required to prove that $AD + BC = 2EF$.

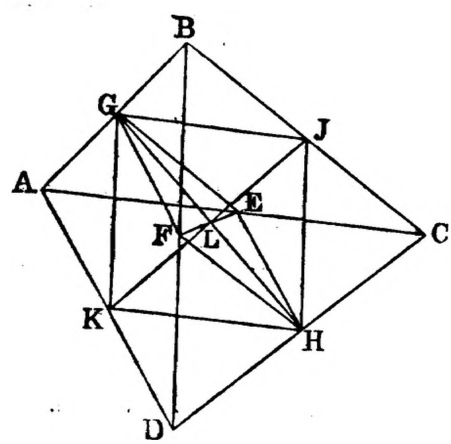
Dem.—Through A draw AH ∥ to DC, meeting EF, BC in G, H.

Now (XXXIV.) $AD = GF$, and $HC = GF$; ∴ $AD + HC = 2GF$, and (5) $BH = 2EG$. Hence $AD + BC = 2EF$.

8. See "Sequel to Euclid," Prop. III., Cor. 2, p. 3.

9. Let ABCD be a quadrilateral; AC, BD its diagonals. Bisect AC, BD in E, F. Join EF. Bisect AB, CD, BC, AD in G, H, J, K. Join GH, JK. It is required to prove that the lines EF, GH, JK are concurrent.

Dem.—Join EG, EH, FG, FH, GJ, GK, HJ, HK.



Now ((2) and (5)) GF is ∥ to AD, and $= \frac{1}{2}$ AD. Similarly, EH is ∥ to AD, and $= \frac{1}{2}$ AD; hence GF is = and ∥ to EH; ∴ (XXXIII.) GFHE is a ▭; hence (XXXIV., 1) the diagonal EF bisects GH in L. In like manner GJHK is a ▭, and the diagonal JK bisects GH. Hence the lines EF, GH, JK are concurrent.

PROPOSITION XLV.

1. Let A and B be two rectilineal figures. It is required to construct a rectangle equal to the sum of A and B.

Sol.—Construct a rectangular parallelogram EFGH equal to A (XLV.), and to the straight line GH apply a ▭ GHIK equal to B, and having the ∠ GHI a right angle. FI is the required rectangle.

Dem.—The figure FI is equal to the sum of A and B, and it is evidently a rectangle.

2. If we apply the ▭ GHIK to the left of GH, it is evident that EFKI will be the required rectangle.

PROPOSITION XLVI.

1. (1) Let AB, CD be equal lines. Upon AB, CD describe squares ABEF, CDGH. It is required to prove that ABEF = CDGH.

Dem.—Join AE, CG. Now AB = BE, and CD = DG; but AB = CD; hence AB and BE = CD and DG, and the ∠ ABE = CDG; ∴ (IV.) the △ ABE = CDG; but ABEF = 2ABE, and CDGH = 2CDG. Hence ABEF = CDGH.

(2) Let ABEF = CDGH. It is required to prove that AB = CD.

Dem.—If not, from AB cut off AJ = CD; and on AJ describe the square AJKL. Now since AJ = CD, AJKL = CDGH; but CDGH = ABEF (hyp.); ∴ AJKL = ABEF, which is absurd. Hence AB = CD.

2. Let ABCD be a square, and BD one of its diagonals. In BD take a point E, and through E draw FG, HJ ∥ to AB, AD. It is required to prove that HG, FJ are squares.

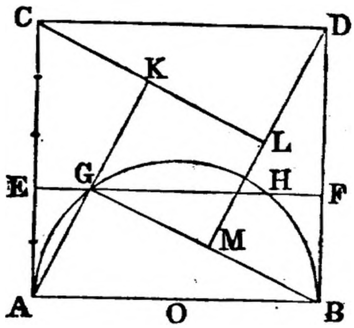
Dem.—The ∠ ADB = ABD (V.); but ADB = HEB (XXIX.); ∴ ABD = HEB; hence the side HE = HB; but HB = EG, and HE = BG; ∴ HB, HE, GB, EG are all equal. Again, the ∠ˢ EHB, GBH equal two right ∠ˢ: but GBH is right; ∴ EHB is right, and (XXXIV.) the opposite ∠ˢ are equal. Hence EGBH is a square. In like manner EJDF is a square.

3. Let ABCD be a square, and E, F, G, H points in the sides AB, BC, CD, DA respectively equidistant from A, B, C, D. Join EF, FG, GH, HJ. It is required to prove that EFGH is a square.

Dem.—The $\triangle^s$ AHE, BEF are equal in every respect (IV.); $\therefore$ the side EH = EF. Similarly, EF = GF, and EH = GH. Hence the four sides are equal. Again, the $\angle$ AHE = BEF. To each add the $\angle$ AEH, and we have the $\angle^s$ AHE, AEH equal to the $\angle^s$ BEF, AEH; but AHE + AEH = a right $\angle$, since the $\angle$ at A is right; $\therefore$ BEF + AEH = a right $\angle$. Hence the $\angle$ FEH is right. In like manner the other $\angle^s$ are right; $\therefore$ EFGH is a square.

4. Let ABCD be a square. It is required to divide it into five equal parts, namely, four right-angled triangles and a square.

Sol.—Divide AC into five equal parts, and let AE = $\frac{2}{5}$ AC. Through E draw EF ∥ to AB. Upon AB describe the semicircle AGHB, cutting EF in the points G, H. Join AG, and produce



it. From C let fall a ⊥ CK on AK, and produce it. Join BG. From D let fall DM ⊥ to BG, meeting CK produced in L. ABCD is divided into five equal parts.

Dem.—Join OG. Because O is the centre of AGHB, OG = OA; $\therefore$ (V.) the $\angle$ OAG = OGA. Similarly, the $\angle$ OBG = OGB. Hence (XXXII., Cor. 7) the $\angle$ AGB is right. Again, since the $\angle$ AKC is right, the $\angle^s$ KCA, KAC are together equal to a right $\angle$, and therefore equal to the $\angle$ CAB, which is right. Reject the $\angle$ KAC, and we have the $\angle$ KCA = KAB, and the $\angle$ CKA = AGB, because each is right, and the side AC = AB; hence (XXVI.) the $\triangle$ AKC = AGB; $\therefore$ AK = BG, and CK = AG. In like manner it can be shown that the $\triangle^s$ CLD, BMD are each equal to AGB. Hence the four $\triangle^s$ are equal, and the lines AK, BG, CL, DM are equal, and also the lines AG, BM, CK, DL; hence the remainders GK, GM, LK, LM, are equal. Again, the rectangle ABEF is $\frac{2}{5}$ ABCD, and the $\triangle$ AGB is $\frac{1}{2}$ ABEF; $\therefore$ AGB is $\frac{1}{5}$ ABCD; $\therefore$ AKC, CLD, BMD are each $\frac{1}{5}$ ABCD. Hence KGML must be $\frac{1}{5}$ ABCD, and it is a square, for we have proved the sides equal, and the $\angle^s$ are right angles.

PROPOSITION XLVII.

1. **Dem.**—ACHK=AOLG; but AOLG is the rectangle AG . AO; that is, AB . AO, and ACHK is AC^2. Hence $AC^2 = AB . AO$. Similarly, $BC^2 = AB . BO$.

2. **Dem.**—From GA cut off GM = GL, and draw MN ∥ to GL. Now the figure AL = AH (XLVII.); but $AH = AC^2 = AO^2 + OC^2$; and $GN = MN^2 = AO^2$; hence $OM = CO^2$; but OM = AO . OB, since ON = OB. Hence $CO^2 = AO . OB$.

3. **Dem.**—$AC^2 = AO^2 + OC^2$, and $BC^2 = BO^2 + OC^2$. Subtracting, we get $AC^2 - BC^2 = AO^2 - BO^2$.

4. Let AB, CD be the lines whose squares are given. It is required to find a line whose square shall be equal to the sum of the squares on AB and CD.

Sol.—Erect AE ⊥ to AB, and make it equal to CD. Join BE. Now (XLVII.) $BE^2 = AB^2 + AE^2 = AB^2 + CD^2$.

5. Let ACB be a △ whose base AB is given, and the difference of the squares of its sides. It is required to prove that the locus of C is a right line ⊥ to AB.

Dem.—From C let fall a ⊥ CO on AB. Now (3) $AC^2 - BC^2 = AO^2 - BO^2$; but $AC^2 - BC^2$ is given; ∴ $AO^2 - BO^2$ is given, and ∴ O is a given point; ∴ the line OC is given in position. Hence OC is the locus of C.

6. **Dem.**—Let P, Q be the points in which AC, GC intersect BK. Now (IV.) the △s CAG, BAK are equal in every respect; ∴ the ∠ ACG = AKB, and the ∠ CPQ = APK (XV.); ∴ (XXXII., Cor. 7) the ∠ CQP = KAP; ∴ CQP is a right ∠, and CG is ⊥ to BK.

7. See "Sequel to Euclid," Book I., Prop. XXIII. (3).

8. **Dem.**—Since EB = AH, AB = AE + AH, and AC is the square on AB; ∴ AC is equal to the square on the sum of AE and AH; but AC exceeds EG by four times the △ AEH, and EG is the square on EH; hence the square on the sum of AE and AH exceeds the square on EH by four times the △ AEH.

9. **Dem.**—Join PH, QC. Now (XXXVII.) the △ PCQ = PBQ. To each add APQ, and we have the △ ACQ = APB. Again, the sum of the △s KAP, HCP equals $\frac{1}{2}$ KC, and the △ KAB = $\frac{1}{2}$ KC (XLI.); ∴ KAB = KAP and HCP. Reject the △ KAP, and we have the △ APB = HCP; but APB = AQC; hence HCP = AQC, and their bases HC, AC are equal. Hence (XL.) their altitudes PQ, PC are equal.

10. See "Sequel to Euclid," Book I., Prop. XXIII. (2).

11. Let M, N be two lines. It is required to find a line whose square shall be equal to $M^2 - N^2$.

Sol.—Draw a line $AB = M$, and in it take $AC = N$. Erect CE ⊥ to AB. With A as centre, and AB as radius, describe a ⊙ cutting CE in D. CD is the required line.

Dem.—Join AD. Now $AD^2 = AC^2 + CD^2$; ∴ $CD^2 = AD^2 - AC^2 = AB^2 - AC^2 = M^2 - N^2$.

12. **Dem.**—From AC cut off $AD = BC$; then, evidently, CD is the difference between AC and CB. On AB describe a square ABFG, and on CD describe a square CDEH; and produce DE, EH to meet ABFG in G, F (figure similar to that on p. 89, "Elements").

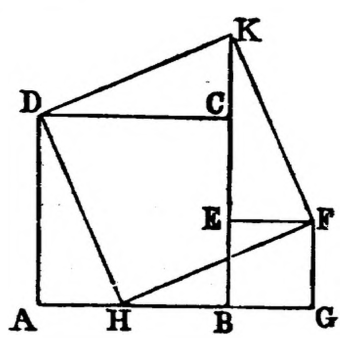
Now CE is less than AF by the sum of the four triangles; that is, by four times the △ ABC. Hence $CD^2 + 4\,ABC = AB^2$.

13. **Dem.**—Join CF, CG, cutting AE, BK in P, Q. Through A draw AM ∥ to GC, cutting BK in R, and meeting LC produced in M. Join BM, cutting AE in N. Now, because AM is ∥ to GC, and AG to ML, AGCM is a ▭; ∴ $AG = CM$; but $AG = BF$; ∴ $BF = CM$; ∴ FCMB is a ▭; ∴ CF is ∥ to BM; hence (XXIX.) the ∠ ANM = APC; but APC is a right ∠ (6); ∴ ANM is right, and AN is ⊥ to BM. In like manner BR is ⊥ to AM; and OM being ⊥ to AB, ∴ AN, BR, OM are the ⊥s of the △ AMB; ∴ (XXXII., Ex. 6) these lines are concurrent; that is, the lines AE, BK, CL are concurrent.

14. Let ABC be an equilateral triangle. Let fall a ⊥ AD on BC.

Dem.—$AB^2 = AD^2 + BD^2$ (XLVII.); ∴ $4\,AB^2 = 4\,AD^2 + 4\,BD^2$, but $AB^2 = 4\,BD^2$, since $AB = BC = 2\,BD$. Subtracting, we get $3\,AB^2 = 4\,AD^2$.

15. **Sol.**—In AB take $AH = BG$. Join DH, FH. These lines



divide the figure into the parts required.

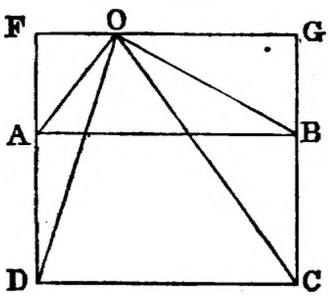
Dem.—For if we take the Δ AHD and place it in the position DCK, and place the Δ FHG in the position FKE, the figure HFKD will be equal to the figure AGFECD; and it is evidently a square.

16. Let AB be the hypotenuse of the right-angled Δ ACB. Bisect BC, AC in D, E. Join AD, BE. It is required to prove that $4\,AD^2 + 4\,BE^2 = 5\,AB^2$.

Dem.—$4\,AD^2 = 4\,AC^2 + 4\,CD^2$; but $BC^2 = 4\,CD^2$; $\therefore$ $4\,AD^2 = 4\,AC^2 + BC^2$. Similarly, $4\,BE^2 = 4\,BC^2 + AC^2$. Adding, we get $4(AD^2 + BE^2) = 5(AC^2 + BC^2) = 5\,AB^2$.

17. Let ABC be a Δ, and O a point within it. Through O draw $\perp^s$ AD, BE, CF to BC, CA, AB. It is required to prove that $AF^2 + BD^2 + CE^2 = BF^2 + DC^2 + EA^2$. Now (2) $AF^2 - BF^2 = AO^2 - BO^2$; $BD^2 - CD^2 = BO^2 - CO^2$; and $CE^2 - AE^2 = CO^2 - OA^2$. Adding, we get $AF^2 + BD^2 + CE^2 - (BF^2 + DC^2 + EA^2) = 0$; and hence $AF^2 + BD^2 + CE^2 = BF^2 + DC^2 + EA^2$. Similarly for a figure of any number of sides.

18. Let ABCD be a rectangle, and O any point. Join OA, OB, OC, OD. It is required to prove that $OA^2 + OC^2 = OB^2 + OD^2$.



Dem.—Produce DA, CB to F, G, and let fall $\perp^s$ OF, OG on DF, CG.

Now, $OD^2 = DF^2 + OF^2$; and $OA^2 = AF^2 + OF^2$; $\therefore$ $OD^2 - OA^2 = DF^2 - AF^2$. Similarly, $OC^2 - OB^2 = CG^2 - GB^2$; but $DF^2 = CG^2$, and $AF^2 = GB^2$; $\therefore$ $OD^2 - OA^2 = OC^2 - OB^2$; and, by transposition, we have $OD^2 + OB^2 = OC^2 + OA^2$.

19. Let AB be the hypotenuse of a right-angled Δ ABC. It is required to divide it into two parts, such that the difference of their squares shall equal AC^2.

Sol.—Bisect BC in D. Join AD, and let fall the $\perp$ DE on AB. $AE^2 - BE^2 = AC^2$.

Dem.—$AD^2 - BD^2 = AE^2 - BE^2$ (3); that is, $AC^2 + CD^2 - BD^2 = AE^2 - BE^2$; but $CD^2 = BD^2$ (const.); $\therefore$ $AC^2 = AE^2 - BE^2$.

20. Let ABC be the Δ. From B, C let fall $\perp^s$ BE, CD on AC, AB. It is required to prove that $AB.BD + AC.CE = BC^2$.

Dem.—On BC describe a square BCFG. Produce BE, CD to H, J; and through B, C draw BL, CK $\parallel$ to DJ, EH, and make BL = AB, and CK = AC. Complete the $\square^s$ BLJD, CKHE. Draw AM $\parallel$ to CF, meeting GF in M. Now it can be shown, as in (XLVII.), that BM = BJ, and CM = CH; $\therefore$ BF = BJ + CH; but $BF = BC^2$, $BJ = AB.BD$, and $CH = AC.CE$. Hence $BC^2 = AB.BD + AC.CE$.

Miscellaneous Exercises on Book I.

1. See "Sequel to Euclid," Book I., Prop. III., Cor. 1.

2. Let DEF be the original Δ, ABC the Δ formed by drawing through each vertex a $\parallel$ to the opposite side. Let fall a $\perp$ FG on DE. It is required to prove that GF bisects BC perpendicularly.

Dem.—The $\angle$ CFG = DGF (XXIX.); but DGF is right; $\therefore$ CFG is right. Again, BF = DE (XXXIV.), and CF = DE; $\therefore$ BF = CF. Hence GF bisects BC perpendicularly. Similarly, the $\perp^s$ from D, E on EF, DF bisect AB, AC perpendicularly.

3. Let ABC be a given $\angle$, and D a given point. It is required to draw a line through D, so that the parts DA, DC, intercepted by AB, BC, may be equal.

Sol.—Through D draw DE $\parallel$ to AB, meeting BC in E, and make EC = BE. Join CD, and produce it to meet AB in A.

Dem.—AC is bisected in D (XL., Ex. 3).

4. Let BD, CE, two of the medians of the Δ ABC, intersect in H. Join AH, and produce it to meet BC in F. It is required to prove that AF is the third median.

Dem.—Produce AF to G; draw BG $\parallel$ to EH, and join GC. Now (XL., Ex. 3) AG is bisected in H; and in the Δ AGC, HD is $\parallel$ to GC (XL., Ex. 2); hence BHCG is a $\square$; and $\therefore$ (XXXIV., Ex. 1) BC is bisected by HG, in F. Hence AF is a median of the Δ ABC.

5. See "Sequel to Euclid," Book I., Prop. IV., Cor.

6. Let a, b be the two sides, and c the median of the third side. It is required to construct a Δ having two sides respectively equal to a and b, and the median of the third side equal to c.

Sol.—Construct the Δ ABC, having AB = a, AC = b, and BC = $2c$. Bisect BC in D. Join AD, and produce it until DE = AD. Join EC. ACE is the required triangle.

Dem.—The Δ^s ADB, CDE are equal (IV.) in every respect;

$\therefore$ AB = CE; but AB = a; $\therefore$ CE = a, and AC = b, and BC = $2c$; $\therefore$ CD = c.

7. (1) See (xx., Ex. 9).

(2) Let a, b, c be the sides of the $\triangle$, and α, β, γ the medians.

Dem.—$\frac{2}{3}\beta + \frac{2}{3}\gamma > a$ (Ex. 5). In like manner $\frac{2}{3}\gamma + \frac{2}{3}\alpha > b$; and $\frac{2}{3}\alpha + \frac{2}{3}\beta > c$. Adding, we have $\frac{4}{3}(\alpha + \beta + \gamma) > (a + b + c)$; and therefore $(\alpha + \beta + \gamma) > \frac{3}{4}(a + b + c)$.

8. Let a be the side, and b, c, the medians. It is required to construct a $\triangle$, having a side equal to a, and the medians of the remaining sides equal to b, c.

Sol.—Construct a $\triangle$ ABC (XXII.), having BC (the base) = a, AB = $\frac{2}{3}b$, and AC = $\frac{2}{3}c$. Bisect BC in D. Join DA, and produce to E, so that AE = 2 AD. BEC is the required $\triangle$.

Dem.—Produce BA, CA to meet CE, BE in F, G. Now ED is a median of the $\triangle$ EBC (const.), $\therefore$ (4) BF, CG are medians; hence (5) BA = $\frac{2}{3}$ BF; but BA = $\frac{2}{3}b$; $\therefore$ BF = b. Similarly, CG = c.

9. Let a, b, c be the medians of a $\triangle$. It is required to construct it.

Sol.—Construct a $\triangle$ ABC, having AB = $\frac{2}{3}a$, BC = $\frac{2}{3}b$, and CA = $\frac{2}{3}c$. Bisect BC in D. Join AD, and produce it to E, so that DE = AD. Produce CB to F, and make BF = BC. Join AF, EF. AFE is the $\triangle$ required.

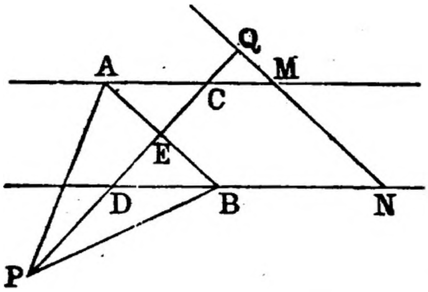
Dem.—Join EB, and produce it to meet AF in H. Produce AB to meet EF in G. Join CE. Now since AD = DE, and BD = CD, ABEC is a parallelogram; $\therefore$ BH is $\parallel$ to AC. Hence (XL., Ex. 3) AF is bisected in H. Similarly, FE is bisected in G, and (const.) AE is bisected in D; $\therefore$ (Def.) AG, DF, EH are the medians; hence (Ex. 5) AB = 2BG; but AB = $\dfrac{2a}{3}$; $\therefore$ AG = a. In like manner it can be shown that FD = b, and EH = c.

10. Let ABC be the $\triangle$. Let fall a $\perp$ AD on BC. Bisect the $\angle$ BAC by AE, meeting BC in E. It is required to prove the $\angle$ DAE = $\frac{1}{2}$ (ACB − ABC).

Dem.—From AB cut off AF = AC. Join CF, cutting AD, AE in G, H. Join EF. Now (V.) the $\angle$ AFC = ACF, and (IV.) the base EC = EF; $\therefore$ the $\angle$ EFC = ECF; hence the $\angle$ AFE = ACE; but AFE = FBE + FEB (XXXII.); $\therefore$ ACB = ABC + FEB; hence FEB = ACB − ABC; but ECF = $\frac{1}{2}$ FEB; $\therefore$ ECF = $\frac{1}{2}$ (ACB − ABC). Again, the $\angle$ AHG is right (IV., Ex. 1), and GDC is right, and the $\angle$ AGH = CGD (XV.); $\therefore$ the $\angle$ GAH = GCD. Hence GAH = $\frac{1}{2}$ (ACB − ABC).

11. Let AM, BN be the two || lines, and P the given point. It is required to find in AM, BN two points equidistant from P, and whose line of connexion shall be || to a given line MN.

Sol.—From P let fall a ⊥ PQ on MN. Bisect the part CD



between AM, BN in E. Through E draw AB || to MN. A, B are the required points.

Dem.—Join AP, BP. Now the ∠ PEB = PQN (xxix.); but PQN is a right ∠, ∴ PEB is right; and since CD is bisected in E, ∴ (xxix., Ex. 4) AB is bisected in E. Now AE = BE, and EP common, and the ∠ AEP = BEP; ∴ (iv.) AP = BP.

12. Let *a* be the side, and *b*, *c* the two diagonals.

Sol.—Construct the △ AEB, having AB = *a*, AE = $\frac{1}{2}$ *b*, and BE = $\frac{1}{2}$ *c*. Produce AE, BE to C, D, so that CE = AE, and DE = BE. Join CD, AD, BC. ABCD is the required parallelogram.

Dem.—The side AB = *a*, and AC, BD = *b*, *c*.

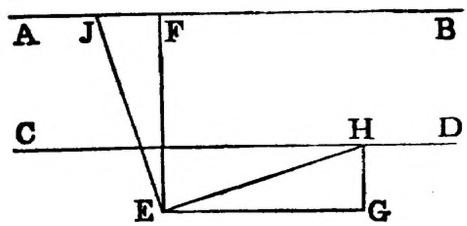
13. Let ABC be a △, having the side AB greater than AC. It is required to prove that BE, the median of AC, is greater than CF, the median of AB.

Dem.—Let BE, CF intersect in G. Join AG, and produce it to meet BC in D. AD is the median of BC. Now because BD = CD, AD common, and the base AB greater than AC, ∴ (xxv.), the ∠ ADB is greater than ADC. Again, BD = CD, GD common, and the ∠ BDG greater than CDG; ∴ (xxiv.) BG is greater than CG; but BG = $\frac{2}{3}$ BE, and CG = $\frac{2}{3}$ CF (5). Hence BE is greater than CF.

14. Let AB, CD be two || lines, and E a given point. It is required to find in AB, CD two points that shall subtend a right angle at E, and be equally distant from it.

Sol.—From E let fall a ⊥ EF on AB. Draw EG || to AB, and make it equal to EF. From G draw GH ⊥ to CD. In AB take FJ = GH. H, J are the required points.

Dem.—Join EH, EJ. Because EF = EG, and FJ = GH, and the ∠ EFJ = EGH, ∴ (IV.) EJ = EH, and the ∠ FEJ = GEH. To each add the ∠ FEH, and we have the ∠ JEH = FEG; but FEG is a right ∠. Hence JEH is right.



15. Let ABC be an isosceles △, and D a point in the base BC. From D let fall ⊥s DE, DF on AB, AC. From B let fall a ⊥ BG on AC. It is required to prove that BG = DE + DF.

Dem.—From D draw DH ∥ to AC, meeting BG in H. Now (XXIX.) the ∠ HDB = ACD; but ACD = ABD (hyp.); ∴ HDB = EBD, and the ∠ BHD = BED, each being right; ∴ (XXVI.) BH = DE; but HG = DF (XXXIV.). Hence BG = DE + DF.

16. See Book IV., Prop. v., Ex. 1, second proof.

17. Let ABC be the △. Bisect the ∠ BAC by AD, meeting BC in D. From D draw DE, DF ∥ to AB, AC. AEDF is an inscribed lozenge.

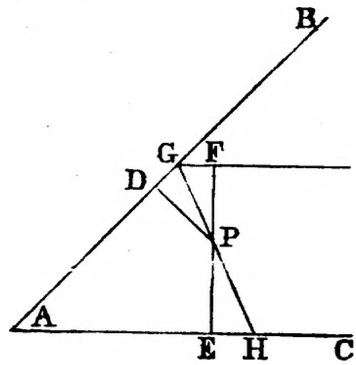
Dem.—The ∠ EAD = ADF (XXIX.); but EAD = FAD (const.); ∴ ADF = FAD, and ∴ AF = DF. Similarly, AE = DE; but (XXXIV.) AF = DE, and AE = DF. Hence the four sides AF, DF, AE, DE are equal; ∴ AEDF is a lozenge.

18. See "Sequel to Euclid," Book I., Prop. XIV.

19. (1) Let AB, AC be two fixed lines, and P the point. Let fall ⊥s PD, PE on AB, AC; then, being given the sum of PD and PE, it is required to find the locus of P.

Dem.—Produce EP to F, and make PF = PD. Through F draw GF ∥ to AC, meeting AB in G. Join GP, and produce it both ways; GP is the required locus. Because PF = PD, to each add PE, and we have EF = PD + PE; ∴ EF is given; and since GF is at a given distance from AC, GF is given in position. Again, since each of the ∠s PFG, PDG is right, $PF^2 + FG^2 = PD^2 + DG^2$; but $PF^2 = PD^2$ (const.); ∴ $GF^2 = GD^2$; ∴ GF = GD. Now GF = GD, GP common, and the base PF = PD; ∴ (VIII.) the ∠ PGF = PGD. Then, since AB, GF are

two fixed lines, and GP bisects the ∠ between them, ∴ GP is given in position, and is the locus of P.



(2) May be done in like manner.

20. Let ABC be an equilateral △, and P any point within it. From P let fall ⊥s PD, PE, PF on AB, BC, CA, and from A let fall a ⊥ AK on BC. It is required to prove that PD + PE + PF = AK.

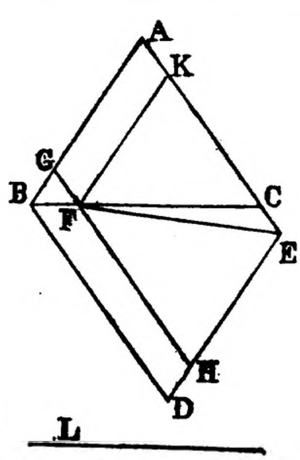
Dem.—Through P draw GH ∥ to BC, meeting AB, AC, AK in G, H, L; and from G let fall a ⊥ GJ on AC. Now the ∠ AGH = ABC (XXIX.); ∴ AGH is an ∠ of an equilateral △. Similarly, AHG is an ∠ of an equilateral △. Hence AGH is an equilateral △; ∴ AL = GJ; but GJ = PD + PF (Ex. 15); ∴ AL = PD + PF, and PE = LK. Hence AK = PD + PE + PF.

21. See "Sequel to Euclid," Book I., Prop. XI.

22. See "Sequel to Euclid," Book I., Prop. XI., Cor. 1.

23. Let ABC be a △, and L a given length. It is required to find a point F in BC, such that if FK, FG be drawn ∥ to AB, AC, the sum of AG, AK shall be equal to L.

Sol.—From B draw BD ∥ to AC, and make it = L. From D

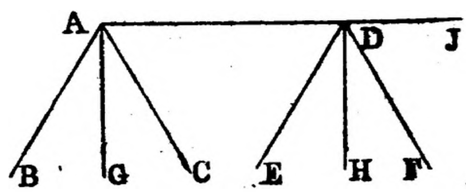


draw DE ∥ to AB, and produce AC to meet it in E. Bisect the ∠ AED by EF, meeting BC in F. F is the point required.

Dem.—Through F draw GH ∥ to BD, and FK ∥ to AB. Now the ∠ HEF = KEF (const.), and (XXIX.) the ∠ KEF = EFH; ∴ EFH = HEF, and ∴ HE = HF; but HE = FK, ∴ FK = FH. To each add FG, and we have FK + FG = GH; that is, AG + AK = GH; but GH = BD = L. Hence AG + AK = L.

24. (1) Let BAC, EDF be two ∠s, whose legs AB, DE, AC, DF are respectively ∥. Bisect BAC, EDF by AG, DH. It is required to prove that AG, DH are ∥.

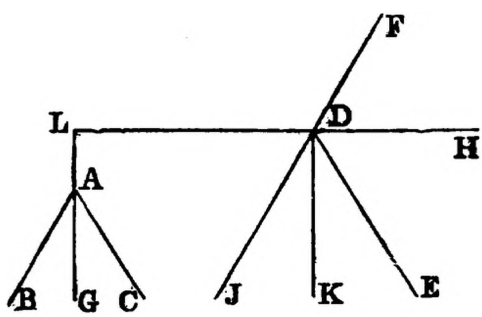
Dem.—Join AD, and produce it to J. Now (XXIX.) the ∠ JDE = JAB, and JDF = JAC; ∴ FDE = CAB; hence FDH = CAG.



And it has been shown that JDF = JAC; ∴ JDH = JAG. Hence (XXVIII.) DH is parallel to AG.

(2) Let BAC, EDF be the ∠s. Bisect BAC, EDF by AG, DH. Produce GA, HD to meet in L. It is required to prove that HL is ⊥ to GL.

Dem.—Produce FD to J, and bisect the ∠ JDE by DK. Now the ∠ FDH = EDH, and JDK = EDK; hence HDK = half sum



of JDE and EDF; but JDE and EDF = two right ∠s; ∴ HDK is a right ∠, and HDK = HLG; ∴ HLG is right. And hence HL is ⊥ to GL.

25. Let ABC be the △ of which A is the vertex; produce BA, CA to D, E. Bisect the ∠s CAD, BAE, by the line FG. From

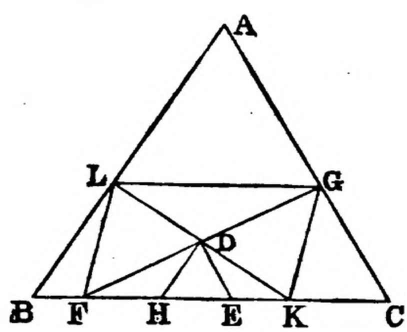
B, C let fall ⊥ˢ BG, CF, on GF. Bisect the ∠ BAC by AH. Join BF, CG. It is required to prove that BF, CG meet on AH.

Dem.—Produce CF to meet AD in D. Now the ∠ CAF = DAF, and CFA = DFA, and AF is common; ∴ (XXVI.) CF = DF; and because the ∠ DFA = HAF, each being right, AH is ∥ to CD. Now, since F is the middle point of the base CD of the △ CBD, and BF joined, and AH ∥ to CD, ∴ (XXVIII., Ex. 7), BF bisects AH. In like manner CG bisects AH. Hence BF, CG meet on AH.

26. **Dem.**—From the vertices A, B, C, of the △ ABC, let fall ⊥ˢ AD, BE, CF on the opposite sides; let them intersect in G. Join DE, EF, FD. It is required to prove that the ⊥ˢ AD, BE, CF bisect the ∠ˢ EDF, DEF, and EFD.

Now the ∠ CDE = CGE (XXXII., Ex. 5), and BDF = BGF; but (XV.) CGE = BGF; ∴ CDE = BDF; and CDA = BDA, since each is right; ∴ EDA = FDA; hence the ∠ EDF is bisected by AD. In like manner the ∠ˢ DEF, EFD are bisected by BE and CF.

27. Let ABC be a given △, and D a given point within it. It is required to inscribe, in ABC, a ▭ whose diagonals shall intersect in D.

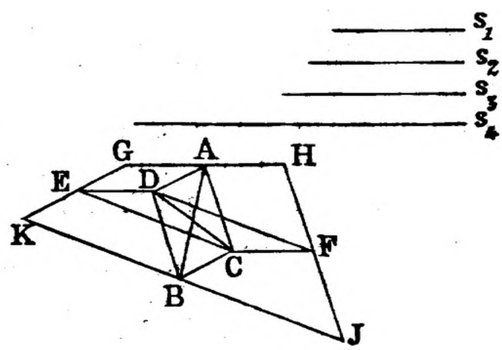


Sol.—Through D draw DE ∥ to AC, and from BE cut off FE = EC. Join FD, and produce it to meet AC in G. Draw DH ∥ to AB; and from HC cut off HK = BH. Join KD, and produce it to meet AB in L. Join GL, FL, GK. GLFK is the required parallelogram.

Dem.—FG is bisected in D (XL., Ex. 3). Similarly, KL is bisected in D. Hence (XXXIV., Cor. 5) GLFK is a parallelogram.

28. Let s_1, s_2, s_3, s_4 be the sides of the quadrilateral; and A, B the middle points of two opposite sides. It is required to construct it.

Sol.—Join AB, and on it describe the $\triangle$ ACB, having BC $= \frac{1}{2} s_1$, and CA $= \frac{1}{2} s_3$. Complete the ▭ ABCD. Join DC; and describe the $\triangle$ CDE, having DE $= \frac{1}{2} s_2$, and CE $= \frac{1}{2} s_4$. Complete the



▭ DECF. Through A, E, B, F draw GH, GK, JK, JH ∥ respectively to DE, BC, CE, CA. GHJK is the required quadrilateral.

Dem.—HF = AC (xxxiv.), and JF = BD; but AC + BD = 2 AC; hence HJ $= s_3$. In like manner GH $= s_2$, GK $= s_1$, and JK $= s_4$.

29. See "Sequel to Euclid," Book I., Prop. VIII.

30. Let ABC be the given rectilineal figure, and O the given point. From O let fall $\perp^s$ on BC, CA, AB; and let them be denoted by p, p_1, p_2; then, if $p + p_1 + p_2$ be given, it is required to prove that the locus of O is a right line.

Dem.—In BC take a part EF, equal to any given line. Join OE, OF. In AC, AB take GH, JK, each equal to EF. Join OG, OH, OJ, OK. Now let EF be denoted by b, and we have $bp = 2\triangle$ OEF (II. I. Cor. 1), and, similarly, for the $\triangle^s$ OGH, OJK. Therefore $b\,(p + p_1 + p_2)$ is equal to twice the sum of the areas of those triangles; but the bases, and sum of the areas, are given. Hence (Ex. 29) the locus of O is a right line.

31. **Dem.**—Through C and B′ draw CD, B′D ∥ to BB′ and BC. Join DC′, cutting BC in E. Now (xxxiv.) BB′ = CD; but BB′ = CC′ (hyp.); ∴ CD = CC′, and CE is common; and the $\angle$ ACB = DCB, because each is equal to ABC; hence (IV.) the $\angle$ CEC′ = CED; ∴ each is a right $\angle$; ∴ (xxix.) B′DE is

right; hence B′C′D is acute; and ∴ (XIX.) B′C′ is greater than B′D; that is, greater than BC.

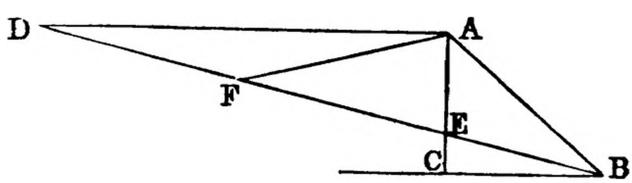
32. (1) **Dem.**—From B let fall a ⊥ BC on L; and produce it to meet AP in Q. In L take any other point S. Join AS, BS, QS. Now, because BCP = QCP, and the ∠ BPC = QPC, and CP common, ∴ (XXVI.) BP = QP. Similarly, BS = QS. Hence AS − SQ = AS − SB; but AS − SQ is less than AQ; ∴ AS − SB is less than AQ; that is, less than AP − BP.

(2) See "Sequel to Euclid," Book I., Prop. XXI.

33. Let ABCD be a quadrilateral. It is required to bisect it by a line drawn from A, one of its angular points.

Dem.—Join AC. Produce DC to E. Through B draw BE ∥ to AC. Join AE. Bisect DE in F. Join AF. AF bisects ABCD. Now the △ AEC = ABC (XXXVII.) To each add the △ ACD, and we have the △ AED = the quadrilateral ABCD; but AED = 2ADF (XXXVIII.); ∴ ABCD = 2ADF.

34. **Dem.**—Bisect ED in F. Join AF. Now (XII., Ex. 2), the lines EF, AF, DF are equal; hence the ∠ FAD = FDA;



but (XXXII.) the ∠ AFE = FAD + FDA; ∴ AFE = 2FDA, and ∴ (XXIX.) = 2DBC; but AF = AB, because each is equal to $\frac{1}{2}$ ED; ∴ the ∠ ABF = AFB; but AFB = 2DBC. Hence ABF = 2DBC.

35. **Dem.**—The three ∠s ABC, BCA, CAB are equal to two right ∠s; ∴ ABO, BAO, BCO are equal to a right ∠; but BOD = ABO + BAO; ∴ BOD and BCO equal a right ∠; and EOC + BCO equal a right ∠; hence BOD + BCO = EOC + B′CO; ∴ the ∠ BOD = EOC.

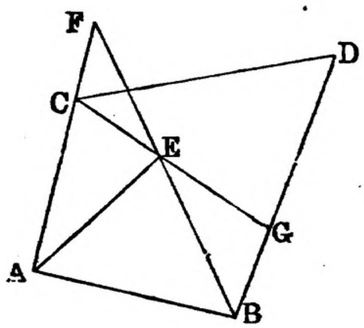
36. The angles of each external triangle are respectively equal to $\frac{1}{2}$(A + B), $\frac{1}{2}$(B + C), $\frac{1}{2}$(A + B). See (XXXII., Ex. 14). Hence the three external triangles are equiangular.

37. (1) **Dem.**—Let ABCD be the quadrilateral. Bisect the ∠s BCD, CDA by CE, DE. It is required to prove that the ∠ CED = $\frac{1}{2}$(DAB + ABC).

Now the $\angle$s DAB, ABC, BCD, CDA are together equal to four right $\angle$s, and the $\angle$s CED, EDC, DCE are equal to two right $\angle$s; hence (CED + EDC + DCE) = $\frac{1}{2}$ (DAB + ABC + BCD + CDA); but EDC = $\frac{1}{2}$ ADC, and DCE = $\frac{1}{2}$ DCB. Hence CED = $\frac{1}{2}$ (DAB + ABC).

(2) Bisect the $\angle$s ABD, ACD by BE, CE. Produce BE, CE to meet AC, BD in F, G. It is required to prove that the $\angle$ CEF = $\frac{1}{2}$ (BAC − BDC).

Dem.—Join AE. Now the $\angle$s of the figure ABEC are equal to



four right $\angle$s; and the $\angle$s of the figure BECD equal to four right $\angle$s; hence the $\angle$s (BAC + ABE + BEC + ACE) = (BEG + GEF + FEC + ECD + CDB + DBE); but ABE = DBE, and ACE = ECD, and BEC = GEF. Reject these, and we have BAC = CDB + GEB + CEF = CDB + 2 CEF. Hence the $\angle$ BAC exceeds CDB by 2 CEF; that is, CEF = $\frac{1}{2}$ (BAC − CDB).

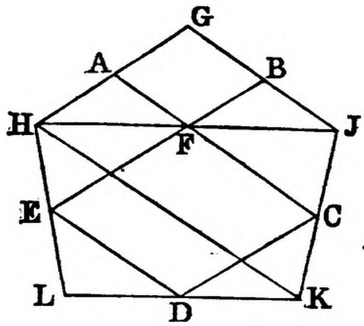
38. **Dem.**—It has been proved (XLVII., Ex. 7) that $EF^2 = AC^2 + 4\ BC^2$. Similarly, $KG^2 = BC^2 + 4\ AC^2$. Adding, we get $EF^2 + KG^2 = 5\ (AC^2 + BC^2) = 5\ AB^2$.

39. Let A, B, C, D, E be the middle points of the sides of a convex polygon of an odd number of sides. It is required to construct it.

Sol.—Join CD, DE; and through C, E draw CF, EF ∥ to DE, CD; and (XXXIV., Ex. 6) construct the △ GHJ, having A, B, F for the middle points of its sides. Join AF, BF, JC, and produce JC to K, so that CK = CJ. Join KD, HE, and produce them to meet in L. GHLKJ is the required polygon.

Dem.—Join HK. Now in the △ HJK, HJ, JK are bisected in F, C; hence (XL., Exercises 2 and 5) FC is ∥ to HK, and

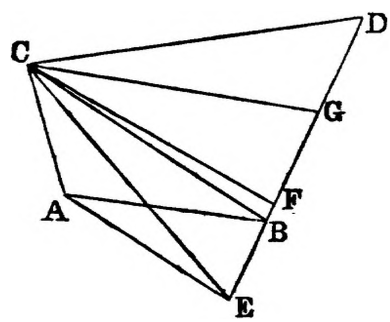
equal to half of it; but FC = ED; ∴ ED is ∥ to HK, and



equal to half HK. And hence (XL., Ex. 3) HL, LK are bisected in E, D.

40. Let ABCD be a quadrilateral. It is required to trisect it by lines drawn from C, one of its angular points.

Sol.—Join BC. Produce DB to E, and draw AE ∥ to BC. Join CE. Trisect ED in F, G (XXXIV., Ex. 3). Join CF, CG. CF, CG trisect the quadrilateral.



Dem.—The △ CEB = CAB (XXXVII.). To each add CBD, and we have the △ CED = the quadrilateral CABD; but the △ CGD = $\frac{1}{3}$ CED; ∴ CGD = $\frac{1}{3}$ CABD. In like manner CFG = $\frac{1}{3}$ CABD.

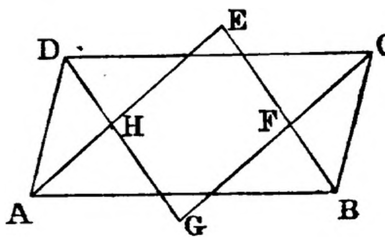
41. Let ABC be a △ whose base BC is given in magnitude and position; and the sum of its sides BA, AC also given. Produce BA to D, and make AD = AC. Bisect the ∠ CAD by AE. Erect CE ⊥ to AC. Join BE, DE; and from E let fall a ⊥ EF on BC produced. It is required to prove that the locus of E is the perpendicular EF.

Dem.—Because AC = AD, and AE common, and the ∠ CAE

= DAE, $\therefore$ (IV.) CE = DE, and the $\angle$ ACE = ADE; but ACE is a right $\angle$ (const.); $\therefore$ ADE is right; hence (XLVII.) $BE^2 - ED^2 = BD^2$; but BD is given, since it is equal to BA + AC; and ED = EC; $\therefore$ $BE^2 - EC^2$ is given, and the base BC is given. Hence (XLVII., Ex. 5) the locus of E is EF, the $\perp$ from E on BC.

42. (1) See XXXII., Ex. 8.

(2) Let ABCD be a parallelogram. It is required to prove that EFGH is a rectangle.



Dem.—The $\angle$s ABC, BAD are together equal to two right $\angle$s (XXIX.); $\therefore$ the $\angle$s EBA, EAB together make a right $\angle$; hence the $\angle$ AEB is right. Similarly, the $\angle$s at F, G, H are right. Hence EFGH is a rectangle.

(3) Let ABCD be a rectangle. It is required to prove that EFGH is a square.

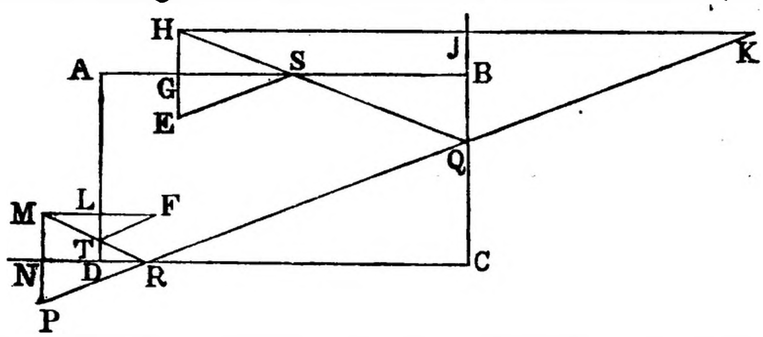
Dem.—Because the $\angle$ BAD = CDA; the $\angle$ BAE = CDG. In like manner the $\angle$ ABE = DCG, and the side AB = CD; $\therefore$ (XXVI.) AE = DG; but AH = DH, since the $\angle$ ADH = DAH; $\therefore$ HE = HG. In like manner all the sides are equal, and the $\angle$s are right $\angle$s. Hence EFGH is a square.

43. **Dem.**—Join AE. Now (XL., Ex. 5) EF = $\frac{1}{2}$ AB = BD; and FG = BD; $\therefore$ EF = FG, and AF = CF (hyp.); $\therefore$ CF and FG = AF, FE; and the $\angle$ CFG = AFE (XV.); hence (IV.) CG = AE; but AE is a median of the $\triangle$ ABC; also CD, a side of the $\triangle$ CDG, is one of the medians of ABC; and BF, the remaining median, is equal to DG (XXXIV.). Hence the sides of the $\triangle$ CDG are equal to the medians of ABC.

44. Let ABCD be the billiard table, E the point from which the ball starts, and F the point through which it will pass.

Sol.—From E let fall a $\perp$ EG on AB; produce EG to H, so that GH = EG. From H let fall a $\perp$ HJ on CB produced; and produce HJ to K, so that JK = HJ. From F let fall a $\perp$ FL on AD, and produce to M, so that LM = LF; and from M let fall a

⊥ MN on CD produced, and produce to P, so that NP = MN. Join KP, intersecting BC in Q and CD in R. Join HQ, MR, inter-

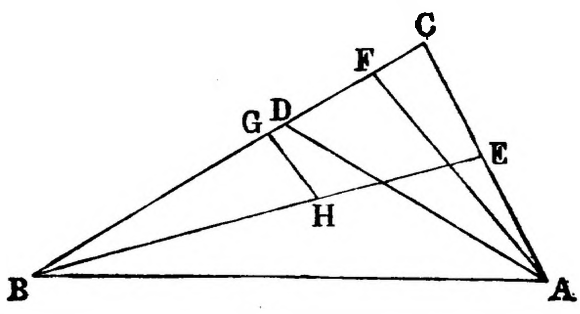


secting AB in S, and AD in T. Join ES, FT. ESQRTF will be the path of the ball.

Dem.—Because EG = HG, GS common, and the ∠ EGS = HGS, ∴ the ∠ ESG = HSG; but HSG = BSQ (xv.); ∴ ESG = BSQ; hence the ball will be reflected in the direction SQ. In like manner it can be shown that the ∠ HQJ = RQC, and therefore the ball will be reflected from Q in the direction QR. Similarly, it will be reflected from R to RT, and from T to TF.

45. Let ABC be the Δ, AD, BE the bisectors of the ∠ˢ A, B. It is required to prove, if AD = BE, that the ∠ CAB = ABC.

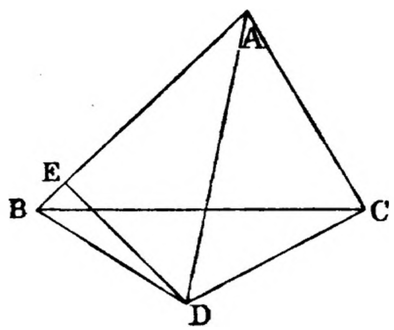
Dem.—If the angle CAB be not equal to ABC, let CAB be the greater; then, since the ∠ CAB is greater than ABC, its



half, the ∠ DAC, is greater than EBC, the half of ABC; then make DAF equal to EBC. Now, since the ∠ DAB is greater than ABE, the whole ∠ FAB is greater than FBA; ∴ the side FB is greater than FA. Cut off BG = FA, and draw GH parallel to FA; then the Δˢ GBH, FAD have evidently two angles in one respectively equal to two angles in the other, and the side BG = AF. Hence BH is equal to AD; but BE is = AD (hyp.). Hence BH = BE, which is absurd. Hence the angle CAB is not unequal to ABC; that is, it is equal to it, and ∴ (vi.) the Δ ABC is isosceles.

46. Let ABC be a Δ, whose base and difference of sides are given. Bisect the $\angle$ BAC by AD. Erect CD $\perp$ to AC. The locus of D is a right line.

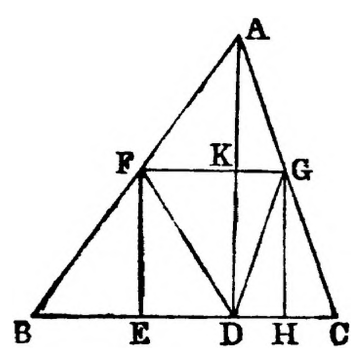
Dem.—Let fall a $\perp$ DE on AB. Join BD. Now (I. XXVI.) the Δs ACD, AED are equal in every respect; $\therefore$ DC = DE, and AC = AE; $\therefore$ AB − AC = BE; but AB − AC is given; $\therefore$ BE is given. Again, $BD^2 - DE^2 = BE^2$; that is, $BD^2 - CD^2 = BE^2$;



hence $BD^2 - CD^2$ is given, and the base BC is given. Now we are given the base, and the difference of the squares of the sides of the Δ BCD. Hence (XLVII., Ex. 5) the locus of the vertex D is a right line perpendicular to the base.

47. Let EFGH be a square inscribed in the Δ ABC. It is required to prove that (BC + AD) s = 2 Δ ABC, where s denotes the side of the square.

Dem.—Let fall a $\perp$ AD on BC. Join DF, DG. Now BD.EF = 2 Δ BFD (II. I., Cor. 1); that is, BD. s = 2 Δ BFD. Similarly, DC . s = 2 Δ DGC; $\therefore$ BC . s = 2 Δ BFD + 2 Δ DGC.

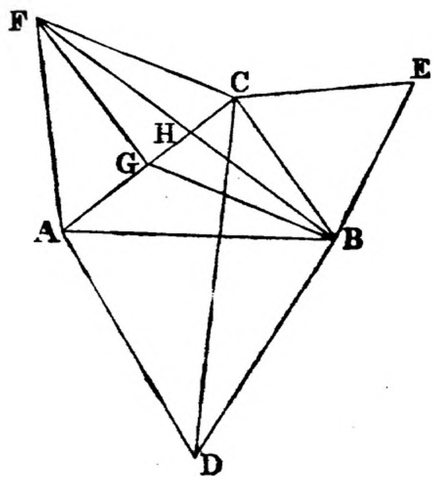


Again, AD. FK = 2 Δ AFD, and AD.GK = 2 Δ AGD; $\therefore$ AD . s = 2 AFDG. Adding, we get (BC + AD) s = 2 Δ ABC.

48. **Dem.**—Let fall a ⊥ CE on AB. Now (XLVII., Ex. 20) $BC^2 = AB.BE + AC.CD$; but (XXVI.) the △s BEC, BDC are equal; since the △ ABC is isosceles; ∴ BE = DC, and AB = AC. Hence $BC^2 = 2\,AC\,.\,CD$.

49. Let ABC be a right-angled △, and let equilateral △s be described on its three sides. It is required to prove that the △ ABD is equal to the sum of the △s ACF, BCE.

Dem.—Bisect AC in G. Join FG, BG, FB, CD. Now the ∠ CAF = BAD; to each add CAB, and we have the ∠ FAB = CAD, and AF = AC, and AB = AD; ∴ (IV.) the △s AFB, ACD are equal. Again, because each of the ∠s FGC, ACB is right, BC, FG are parallel; ∴ (XXXVII.) the △ FGC = FGB. To



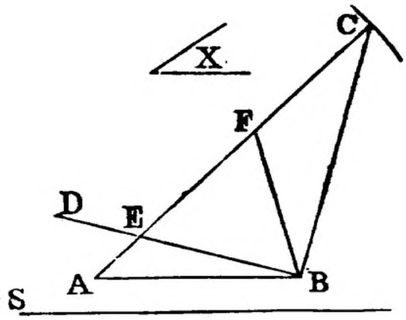
each add the △ FGA, and we have AFC = to the quadrilateral AFBG. Again, to each add the △ AGB, which is ½ ACB, and we have AFC + ½ ACB = AFB. Hence ACD = AFC + ½ ACB. Similarly BCD = BEC + ½ ACB. Add, and we have ACBD = AFC + ACB + BEC. Reject the right angled △ ACB, which is common, and the △ ABD = AFC + BEC.

50. (1) Let AB be the base, X the difference of the base ∠s, and S the sum of the sides. It is required to construct the triangle.

Sol.—Draw BD, making the ∠ ABD = ½ X, and draw BC ⊥ to BD. With A as centre, and a radius equal to S, describe a ⊙, cutting BC in C. Join AC, cutting BD in E. Bisect CE in F. Join BF. AFB is the required triangle.

Dem.—The lines BF, CF, EF are equal (XII., Ex. 2); ∴ FE

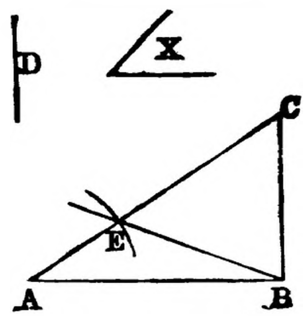
= FB; ∴ the ∠ FBE = FEB; but FEB = FAB + ABE (XXXII.); ∴ FBE = FAB + ABE; hence the ∠ FBA = FAB + 2 ABE;



and hence the ∠ ABE is half the difference of the base ∠s; but ABE = ½ X. Hence the difference of the base ∠s = X; and since FB = FC, AF + FB = AC = S, ∴ the sum of the sides = S.

(2) Let AB be the base, X the difference of the base ∠s, and D the difference of the sides.

Sol.—Draw BE, making the ∠ ABE = ½ X. With A as centre, and a radius equal to D, describe a ⊙, cutting BE in E. Join AE,



and produce it. Draw BC, making the ∠ CBE = CEB, and meeting AE produced in C. ACB is the required triangle.

Dem.—CB = CE (VI.); ∴ AE = AC − CB; but AE = D; ∴ AC − CB = D, and, as before, the difference of the base angles = X.

51. **Sol.**—Let AB be the base, and M the median that bisects the base. To AB apply a ▭ ABCD, whose area is equal to twice the given area (XLV.). Bisect AB in E. With E as centre, and a radius equal to M, describe a ⊙, cutting CD in F. Join AF, BF. AFB is the required △.

52. **Dem.**—Join AG, CG, FG. The △ CED = CGD + CEG, and the △ EBC = BGC − CEG. Subtracting, we get CED − EBC

= 2 CEG. Similarly AED − AEB = 2 AEG. Subtracting, we have AEB + CED − (AED + EBC) = 2 (CEG − AEG). Again, CEG = CFG + EFG, and AEG = AFG − EFG; ∴ CEG − AEG = 2 EFG. And hence 4 EFG = AEB + CED − (AED + EBC).

53. (1) Let ACB be the △. Describe squares AH, AF, CE on the sides AC, AB, BC respectively. Bisect AC in J. Join BJ, EF. It is required to prove that EF = 2 BJ.

Dem.—Produce BJ to M, so that JM = JB, and join MC.

Now (IV.) the △s MJC, AJB are equal in every respect; ∴ MC = AB = BF, and CB = BE; hence MC, CB equal BF, BE. And because AC and BM bisect each other in J, MC and AB are parallel; ∴ the ∠s MCB and ABC are together equal to two right ∠s, and the ∠s EBF, ABC are equal to two right ∠s; since ABF and CBE are right; ∴ the ∠ MCB = EBF; hence (IV.) MB = EF; but MB = 2 BJ; ∴ EF = 2 BJ.

(2) Produce MB to meet EF in N. MN is ⊥ to EF.

Dem.—From the equal triangles CMB, BFE we have the ∠ CMB = BFE, but CMB = ABM; ∴ BFE = ABM. To each add NBF; and we have BFN + NBF = ABM + NBF; but since ABF is right, ABM + NBF equal a right ∠; ∴ BFN + NBF equal a right ∠; and hence the ∠ BNF is right.

BOOK II.

PROPOSITION IV.

1. **Dem.**—$AB^2 = AB \cdot AC + AB \cdot BC$ (II.);

but $AB \cdot AC = AC^2 + AC \cdot CB$ (III.);

and $AB \cdot BC = BC^2 + AC \cdot CB$ (III.);

Therefore $AB \cdot AC + AB \cdot BC = AB^2 + BC^2 + 2AC \cdot CB$;

that is, $AB^2 = AC^2 + BC^2 + 2\,AC \cdot CB$.

2. Let C be the vertical $\angle$ of the right-angled $\triangle$ ABC. From C let fall a $\perp$ CD on AB. It is required to prove that $DC^2 = AD \cdot DB$.

Dem.—$AB^2 = AC^2 + CB^2$ (I. XLVII.); but $AC^2 = AD^2 + DC^2$; and $CB^2 = BD^2 + DC^2$; $\therefore AB^2 = AD^2 + BD^2 + 2\,DC^2$. Again, $AB^2 = AD^2 + DB^2 + 2\,AD \cdot DB$ (IV.). Hence $DC^2 = AD \cdot DB$.

3. Let ABC be the right-angled $\triangle$. In the base AB cut off $AD = AC$, and $BE = BC$. It is required to prove that $ED^2 = 2\,AE \cdot DB$.

Dem.—$AB^2 = AC^2 + CB^2$ (I. XLVII.) $= AD^2 + BE^2$; but $AD^2 = AE^2 + ED^2 + 2\,AE \cdot ED$ (IV.); and $BE^2 = BD^2 + DE^2 + 2\,BD \cdot DE$; $\therefore AB^2 = AE^2 + ED^2 + 2\,AE \cdot ED + BD^2 + DE^2 + 2\,BD \cdot DE$; also $AB^2 = AE^2 + ED^2 + DB^2 + 2AE \cdot ED + 2\,ED \cdot DB + 2\,AE \cdot DB$ (IV., Cor. 3). Hence $ED^2 = 2\,AE \cdot DB$.

4. Let ABC be the right-angled $\triangle$, CD the $\perp$ from the right angle on the base. It is required to prove that $(AB + CD)^2$ exceeds $(AC + CB)^2$ by CD^2.

Dem.—$AC \cdot CB$ is equal to twice the $\triangle$ ACB, and $AB \cdot CD$ is equal to twice the $\triangle$ ACB; $\therefore AC \cdot CB = AB \cdot CD$.

Now $(AB + CD)^2 = AB^2 + CD^2 + 2\,AB \cdot CD$;

and $(AC + CB)^2 = AC^2 + CB^2 + 2\,AC \cdot CB$.

Subtracting, we have $(AB + CD)^2 - (AC + CB)^2 = AB^2 - BC^2$

$- CA^2 + DC^2$; but $AB^2 - BC^2 = AC^2$; $\therefore (AB + CD)^2 - (AC + CB)^2 = AC^2 - AC^2 + DC^2 = DC^2$.

5. Let the sides of the Δ be denoted by a, b, c, c being t hypotenuse. It is required to prove that $(a + b + c) = 2(c + a)(c + b)$.

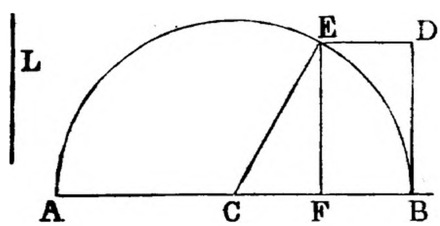
Dem.—$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$; but $(a^2 + b^2 = c^2$ (I. XLVII.); $\therefore (a^2 + b^2 + c^2) = 2c^2$. Hence $(a + b + = 2(c^2 + ac + bc + ab) = 2(c + a)(c + b)$.

PROPOSITION V.

1. Let AB be the given straight line. Bisect it in C. It is required to prove that AC . CB is a maximum.

Dem.—Take any other point D in AB; then $AD . DB + CD^2 = CB^2$ (V.); but $CB^2 = AC . CB$; $\therefore AC . CB = AD . DB + CD^2$; that is, AC . CB is greater than AD . DB by CD^2. Hence, when a line is bisected, the rectangle contained by the parts is a maximum.

2. Let AB be the given straight line, and L the line whose square is given. It is required to divide AB, so that the rectangle contained by its segments will be equal to L^2.



Sol.—Bisect AB in C; with C as centre, and CB as radius, describe a semicircle. Draw BD $\perp$ to AB, and = to L. Through D draw DE $\parallel$ to AB, cutting the semicircle in E; let fall a $\perp$ EF on AB. The rectangle $AF . FB = L^2$.

Dem.—Join CE. Now $AF . FB + CF^2 = CB^2$ (V.) $= CE^2 = CF^2 + FE^2$ (I. XLVII.). Take away CF^2, which is common, and $AF . FB = FE^2 = BD^2 = L^2$.

3. Let ABC be the Δ. From C let fall a $\perp$ CD on AB. It is required to prove that $(AC + BC)(AC - BC) = AB(AD - DB)$.

Dem.—$AC^2 = AD^2 + DC^2$ (I. XLVII.); and $BC^2 = BD^2 + DC^2$. Subtracting, we get $AC^2 - BC^2 = AD^2 - DB^2$; that is (AC

$+ BC)(AC - BC) = (AD + DB)(AD - DB) = AB(AD - DB)$.

4. **Dem.**—$(AC + BC)(AC - BC) = AB(AD - DB)$ (Ex. 3); but $(AC + BC)$ is greater than AB (I. xx.); $\therefore (AC - BC)$ is less than $(AD - DB)$.

5. Let ABC be an isosceles Δ. From C let fall a $\perp$ CE on AB. In AB take any other point D. Join CD. It is required to prove that $CB^2 - CD^2 = AD \cdot DB$.

Dem.—$AD \cdot DB + ED^2 = EB^2$, and $EC^2 = ED^2$. Add together, and we get $AD \cdot DB + CD^2 = CB^2$; $\therefore AD \cdot DB = CB^2 - CD^2$.

6. Let ABC be the Δ. It is required to prove that $AC^2 = (AB + BC)(AB - BC)$.

Dem.—$AC^2 + BC^2 = AB^2$; $\therefore AC^2 = AB^2 - BC^2 = (AB + BC)(AB - BC)$.

PROPOSITION VI.

1. Let AD be the straight line which is bisected in C, and divided unequally in B. It is required to prove Prop. VI. by Prop. V., by producing the line DA in the opposite direction.

Dem.—Produce DA to O, and make $OA = BD$.

Now $OB \cdot BD + CB^2 = CD^2$ (V.); but since $OA = BD$, $OB = AD$. Therefore $AD \cdot DB + CB^2 = CD^2$.

2. Let AB be the given line. It is required to divide it externally in E, so that $AE \cdot EB = L^2$, L being a given line.

Sol.—Bisect AB in C (VI.). Erect BD $\perp$ to AB, and make it equal to L. Join CD. With C as centre, and CD as radius, describe a circle, meeting AB in E. E is the point required.

Dem.—Now $AE \cdot EB + CB^2 = CE^2 = CD^2 = CB^2 + BD^2$. Reject CB^2, which is common, and $AE \cdot EB = BD^2 = L^2$.

3. See Ex. 2.

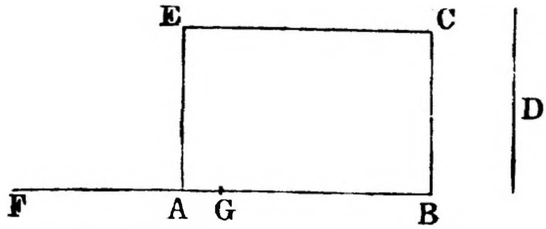
4. Let AD, DB be two lines. Bisect AB in C.

Dem.—Because AB is the sum, CB is half sum; and $AD = AC + CD$, and $DB = CB - CD$; $\therefore AD - DB = 2\,CD$; hence CD is half difference. Now $AD \cdot DB + CD^2 = CB^2$ (V.); $\therefore AD \cdot DB = CB^2 - CD^2$.

5. **Dem.**—Let AB be the sum, and D^2 the difference of their squares. To AB apply the rectangular ▭ $ABCE = D^2$. Now, since the sum multiplied by the difference is equal to the difference of the squares, and that AB is the sum, therefore AE must be the difference. Produce BA to F, and make $AF = AE$. Therefore,

since the sum together with the difference is equal to twice the greater; therefore if we bisect BF in G, BG will be the greater, and AG the less.

If we take AE equal to the difference, and apply the rectangular $\square$ ABCE $= D^2$, we have the second case.



6. See "Sequel to Euclid," Book II., Prop. I., Cor.

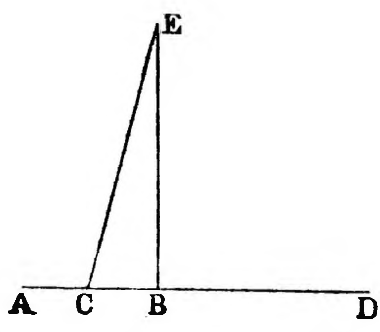
7. The rectangle contained by two straight lines, together with the square described on half their difference, is equal to the square on half their sum.

PROPOSITION VIII.

1. **Dem.**—By the third proof of Prop. VIII. $(AB + BO)^2 = 4\,AB \cdot BO + AO^2$; but $AB \cdot BO = BC^2$ (I. XLVII., Ex. 1), and $AO^2 = AC^2 - CO^2$; $\therefore (AB + BO)^2 = 4\,BC^2 + AC^2 - CO^2$; but $4\,BC^2 + AC^2 = EF^2$ (I. XLVII., Ex. 7); $\therefore (AB + BO)^2 = EF^2 - CO^2$.

2. **Dem.**—$GK^2 = 4\,AC^2 + BC^2$ (I. XLVII., Ex. 7), and $EF^2 = 4\,BC^2 + AC^2$; $\therefore GK^2 - EF^2 = 3\,AC^2 - 3\,BC^2$; but (I. XLVII., Ex. 1) $AC^2 = AB \cdot AO$, and $BC^2 = AB \cdot BO$; $\therefore GK^2 - EF^2 = 3\,(AB \cdot AO - AB \cdot BO) = 3\,AB\,(AO - BO)$.

3. **Sol.**—Let AB be the difference of the lines. Bisect AB in C; erect BE $\perp$ to AB, and make it equal $2\,AB = 2\,R$. Join CE, and produce CB to D. Cut off CD = CE. AD, DB are the required lines.



Dem.—$AD \cdot DB + CB^2 = CD^2$ (VI.) $= CE^2 = CB^2 + BE^2$.

Reject CB^2, which is common, and we have $AD . DB = BE^2 = 4 R^2$. Hence AD, BD are the required lines; for their difference is AB, that is, R, and their rectangle is equal to $4 R^2$.

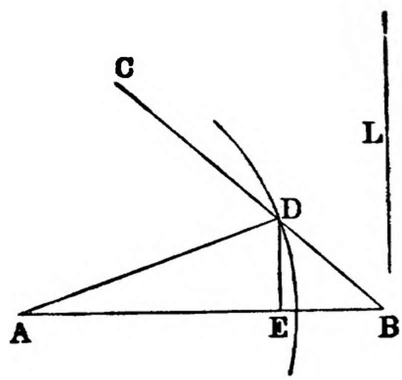
PROPOSITION IX.

1. Let AB be the given line. Bisect it in C. It is required to prove that $AC^2 + CB^2$ is a minimum.

Dem.—Take any other point D in AB. Now $AD^2 + DB^2 = 2 AC^2 + 2 CD^2$ (IX.) $= AC^2 + CB^2 + 2 CD^2$; therefore $AC^2 + CB^2$ is less than $AD^2 + DB^2$ by $2 CD^2$. Hence, when a line is bisected, the sum of the squares on its segments is a minimum.

2. Let AB be a given line. It is required to divide it internally, so that the sum of the squares on the parts may be equal to L^2.

Sol.—Draw BC, making the $\angle$ ABC half a right $\angle$. With A as centre, and a radius equal to L, describe a $\odot$, cutting BC in D. From D let fall a $\perp$ DE on AB. E is the point required.



Dem.—Because the $\angle$ EBD is half a right $\angle$, and the $\angle$ BED right, the $\angle$ BDE is half a right $\angle$; $\therefore$ EB = ED; $\therefore EB^2 = ED^2$; $\therefore AE^2 + ED^2$, that is, AD^2, that is $L^2 = AE^2 + EB^2$. If the circle does not meet the line BC, the question is impossible.

3. **Dem.**—From AC cut off AE = DB. Now $AD^2 + AE^2 = 2 AD . AE + ED^2$ (VII.); that is, $AD^2 + DB^2 = 2 AD . DB + 4 CD^2$.

4. Let AEB be the Δ. In AB take any point D. Join ED. It is required to prove that $2\ ED^2 = AD^2 + DB^2$. From E let fall a $\perp$ EC on AB. Now $AD^2 + DB^2 = 2\ AC^2 + 2\ CD^2$ (IX.); but AC = CE. Therefore $AD^2 + DB^2 = 2\ EC^2 + 2\ CD^2 = 2\ ED^2$.

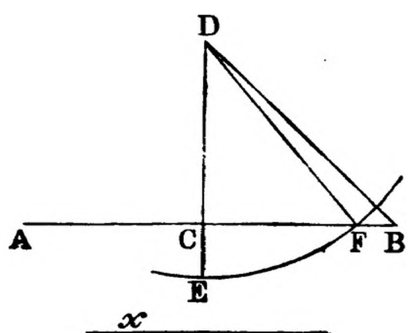
5. See "Sequel to Euclid," Book II., Prop. XII.

PROPOSITION X.

1. (1) Let AB be the sum of the lines, and $2\ X^2$ the sum of the squares.

Sol.—Bisect AB in C. Erect CD $\perp$ to AB, and make it equal to AC or CB. Produce DC to E. Cut off DE = X. With D as centre and DE as radius, describe a $\odot$, cutting AB in F. AF and FB are the required lines.

Dem.—Join DF, DB. Now $AF^2 + FB^2 = 2\ AC^2 + 2\ CF^2$ (IX.) $= 2\ DC^2 + 2\ CF^2 = 2\ DF^2 = 2\ DE^2 = 2\ X^2$.



(2) Let AB be the difference, and $2\ X^2$ the sum of the squares.

Sol.—Bisect AB in C, and erect CD $\perp$ to AB, and make it equal to AC or CB. Produce DC to E. Cut off DE = X. With D as centre, and DE as radius, describe a $\odot$, cutting AB produced in F. AF and FB are the required lines.

Dem.—Join DB, DF. Now $AF^2 + FB^2 = 2\ AC^2 + 2\ CF^2 = 2\ DC^2 + 2\ CF^2 = 2\ DF^2 = 2\ DE^2 = 2\ X^2$.

2. Let CE be the median which bisects the base AB. It is required to prove that $AC^2 + CB^2 = 2\ AE^2 + 2\ CE^2$.

Dem.—From C let fall a $\perp$ CD on AB. Now $AD^2 + DB^2 = 2\ AE^2 + 2\ ED^2$ (IX.), and $CD^2 + CD^2 = 2\ CD^2$. Add, and we get $AC^2 + CB^2 = 2\ AE^2 + 2\ CE^2$.

3. Let BC be the given base of a Δ ABC, the sum of the squares of whose sides AB, AC, is equal to a given square. It is required to prove that the locus of the vertex A is a circle.

Dem.—Bisect BC in D. Join AD. Now (Ex. 2), $BA^2 + AC^2 = 2\,BD^2 + 2\,DA^2$; but $BA^2 + AC^2$ is given (hyp.); $\therefore\ 2\,BD^2 + 2\,DA^2$ is given, and $2\,BD^2$ is given, since BD is half of the given base BC, $\therefore\ 2\,DA^2$ is given; $\therefore$ DA is given, and the point D is given. Hence the locus of A is a circle, having D as centre, and DA as radius.

4. **Dem.**—Bisect AD in E. Join BE, CE. Now (Ex. 2) $AB^2 + BD^2 = 2\,AE^2 + 2\,BE^2$, and $AC^2 + CD^2 = 2\,AE^2 + 2\,CE^2$; but $AB^2 + BD^2 = AC^2 + CD^2$ (hyp.); hence $2\,AE^2 + 2\,BE^2 = 2\,AE^2 + 2\,CE^2$, and therefore $2\,BE^2 = 2\,CE^2$; $\therefore$ BE = CE.

5. See "Sequel to Euclid," Book II., Prop. III.

PROPOSITION XI.

1. Let AB be the line. It is required to cut it externally in extreme and mean ratio.

Sol.—Erect BC $\perp$ to and equal to AB. Bisect AB in D. Join DC. Produce AB to E. Cut off DE = DC. AB is cut in E in extreme and mean ratio.

Dem.—$AE\,.\,EB + DB^2 = DE^2$ (VI.) $= DC^2 = DB^2 + BC^2$. Reject DB^2, which is common, and $AE\,.\,EB = BC^2 = AB^2$.

2. Let AB be a line divided in extreme and mean ratio at C. It is required to prove that $AC^2 - CB^2 = AC\,.\,CB$.

Dem.—$AB\,.\,BC = AC^2$ (hyp.); but $AB = AC + CB$; $\therefore\ (AC + CB)\,CB = AC^2$; that is, $AC\,.\,CB + CB^2 = AC^2$; and therefore $AC\,.\,CB = AC^2 - CB^2$.

3. Let ACB be a right-angled Δ, having $AC^2 = AB\,.\,BC$. From C let fall a $\perp$ CD on AB. It is required to prove that $AB\,.\,BD = AD^2$.

Dem.—$AC^2 = AB\,.\,BC$ (hyp.), and $AC^2 = AB\,.\,AD$ (I. XLVII., Ex. 1); $\therefore\ AD = BC$; $\therefore\ AD^2 = BC^2$; but $BC^2 = AB\,.\,BD$ (I. XLVII., Ex. 1). Hence $AB\,.\,BD = AD^2$.

4. (1). **Dem.**—$AB^2 + BC^2 = 2\,AB\,.\,BC + AC^2$ (VII.); but $AB\,.\,BC = AC^2$ (hyp.) Hence $AB^2 + BC^2 = 3\,AC^2$.

(2) **Dem.**—$(AB + BC)^2 = 4\,AB\,.\,BC + AC^2$ (VIII.); but $AB\,.\,BC = AC^2$ (hyp.) Hence $(AB + BC)^2 = 5\,AC^2$.

*5. **Dem.**—Join FK, AD. Now the square AFGH is double of the △ AFK (I. XLI.) And the rectangle HBDK is double of AKD; but AFGH = HBDK (XI.); ∴ the △ AFK = AKD; and hence (I. XXXIX.) AK is parallel to FD. In like manner, by joining BF, GD, it can be shown that GB is parallel to FD. Hence the three lines AK, FD, GB are parallel.

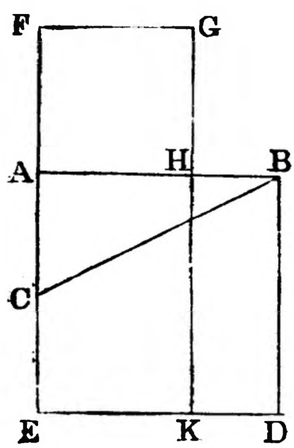
6. **Dem.**—Join BF, and produce CH to meet it in L.

Because EB = EF, the ∠ EBF = EFB, and the ∠s at L are right (XI., Ex. 7); ∴ the ∠ BOL = FCL; but BOL = EOC; ∴ EOC = ECO, and ∴ EC = EO; but EC = EA; ∴ EO = EA; ∴ the ∠ EOA = EAO, and EOC = ECO. Hence the ∠ AOC = OAC + OCA, and is therefore (I. XXXII., Cor. 7) a right angle.

7. Let CH be produced to meet BF at L. It is required to prove that CH is perpendicular to BF.

Dem.—The △s FAB, HAC, are equal (I. IV.) in every respect; ∴ the ∠ FBA = HCA, and the ∠ LHB = AHC (I. XV.); ∴ the ∠ HLB = HAC (I. XXXII., Cor. 2); but HAC is a right angle. Hence HLB is right.

8. **Dem.**—In AB take AH = BC − AC. Produce CA to F, so that AF = AH, then evidently CF = CB. Complete the square AFGH. Produce AC to E, and make CE = AC, and complete

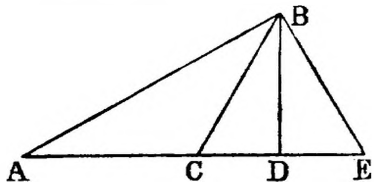


the square ABDE. Produce GH to meet ED in K. Now we have the construction as in Prop. XI., and ∴ AB . BH = AH². Hence AB is divided in "extreme and mean ratio" at H.

* See diagram in Euclid [II. XI.] for this and the two following Exercises.

PROPOSITION XII.

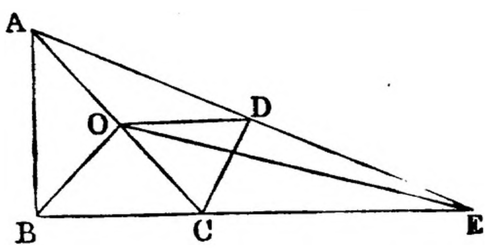
1. **Dem.**—Produce AC, and let fall a $\perp$ BD on AC produced. Make $DE = CD$, and join BE. Now the $\triangle^s$ BCD, BED are equal in every reepect (I. IV.); $\therefore$ the $\angle$ BCE $=$ BEC. And



since the $\angle$ ACB is twice an $\angle$ of an equilateral $\triangle$, each of the $\angle^s$ BCE, BEC is an $\angle$ of an equilateral $\triangle$; hence the $\triangle$ BCE is equilateral; $\therefore$ $BC = CE = 2\,CD$.

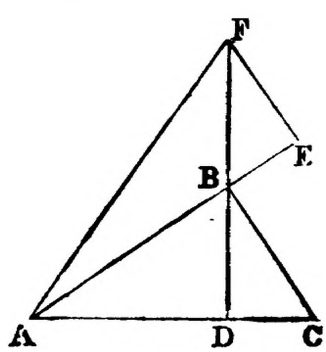
Again, $AB^2 = AC^2 + CB^2 + 2\,AC \cdot CD$; but we have shown that $BC = 2\,CD$. Hence $AB^2 = AC^2 + CB^2 + AC \cdot CB$.

2. **Dem.**—Join AC; bisect it in O. Join BO, DO, EO. Now the lines AO, BO, CO are equal (I. XII., Ex. 2); hence OBC is



an isosceles $\triangle$; $\therefore$ $OE^2 - OC^2 = BE \cdot CE$ (VI., Ex. 6). In like manner $OE^2 - OD^2 = AE \cdot DE$; but $OC = OD$. Hence $AE \cdot DE = BE \cdot CE$.

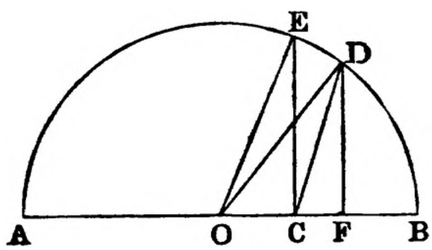
3. **Dem.**—Produce AB, DB. Cut off $BE = DC$, and $BF = BC$.



Join AF, FE. Now the $\angle$ FBC = BDC + BCD (I. xxxii.); but EBC and BDC are right angles; $\therefore$ FBE = BCD; hence (I. iv.) the $\triangle$s BEF and BDC are equal in every respect; therefore the $\angle$ BEF = BDC; hence the $\angle$ BEF is right. Now $AF^2 = AB^2 + BF^2 + 2\,AB.BE$ (xii.), and $AF^2 = AB^2 + BF^2 + 2\,BF.BD$; $\therefore AB.BE = BF.BD$; but BE = DC, and BF = BC. Hence $AB.DC = BD.BC$.

4. **Dem.**—Erect BD $\perp$ to AB, and equal BC. Join AD, CD. Now $AD^2 = AC^2 + CD^2 + 2\,AC.CB$ (xii.), and $CD^2 = CB^2 + BD^2 = 2\,CB^2 = AC^2$; $\therefore AC^2 + CD^2 = 2\,AC^2$; $\therefore AD^2 = 2\,AC^2 + 2\,AC.CB = 2\,AC\,(AC + CB) = 2\,AC.AB$. Again, $AD^2 = AB^2 + BD^2$; but BD = BC; $\therefore AD^2 = AB^2 + BC^2$. Hence $AB^2 + BC^2 = 2\,AC.AB$.

5. **Sol.**—Bisect AB in O. From D let fall a $\perp$ DF on AB. Divide OF in C, so that $2\,OC.CF = p^2$ (the given square). C is the point required.



Dem.—Erect CE $\perp$ to AB. Join OE, OD, CD.

Now $OD^2 = OC^2 + CD^2 + 2\,OC.CF$ (xii.) $= OC^2 + CD^2 + p^2$; but OE = OD; $\therefore OE^2 = OC^2 + CD^2 + p^2$; that is, $OC^2 + CE^2 = OC^2 + CD^2 + p^2$; $\therefore CE^2 - CD^2 = p^2$.

6. **Dem.**—$AD.DB = CD^2 - CB^2$ (vi., Ex. 6); but $CD^2 = 2\,AB^2$ (hyp.); $\therefore AD.DB = 2\,AB^2 - CB^2 = 2\,AB^2 - AB^2 = AB^2$.

PROPOSITION XIII.

1. **Dem.**—From the vertex A let fall a $\perp$ AD on BC. From DB cut off DE = DC. Join AE. Now the $\triangle$ ACD = AED in every respect (I. iv.); $\therefore$ AC = AE, and the $\angle$ AEC = ACE; hence AEC is an $\angle$ of an equilateral $\triangle$; $\therefore$ the $\triangle$ ACE is equilateral; $\therefore$ AC = CE = 2 CD. Again, $AB^2 = BC^2 + CA^2 - 2\,BC.CD$ (xiii.); but we have shown that 2 CD = AC. Hence $AB^2 = BC^2 + CA^2 - BC.AC$.

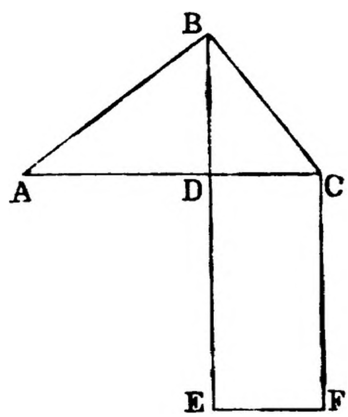
2. See "Sequel to Euclid," Book II., Prop. IV.

3. **Sol.**—Erect BD $\perp$ to and equal to AB. Join AD. Produce AB to C. Cut off AC = AD. C is the point required.

Dem.—$AD^2 = AB^2 + BD^2 = 2\,AB^2$; $\therefore$ $AC^2 = 2\,AB^2$. To each add BC^2, and we have $2\,AB^2 + BC^2 = AC^2 + BC^2 = 2\,AC.BC + AB^2$ (VII.); $\therefore$ $AB^2 + BC^2 = 2\,AC.BC$.

PROPOSITION XIV.

1. **Sol.**—Let a line CD be found (XIV.) whose square is equal to the given difference of squares. On CD construct a rectangle CE, equal to the given rectangle. Produce CD to A, so that $CA.AD = DE^2$ (VI., Ex. 2). Produce ED. From A inflect AB = DE to the line DB, and join BC. BC and BD are the required lines.



Dem.—Because $AB^2 = DE^2 = CA.AD$, the $\angle$ ABC is right (I. XLVII., Ex. 1); $\therefore$ $AB.DC = BD.BC$ (XII., Ex. 3); hence the rectangle CE = BD . BC, and CE is equal to the given rectangle. Also because the $\angle$ BDC is right, $BD^2 - BC^2 = DC^2$, which is equal to the given difference of squares.

2. See Book II., Ex. 6, Miscellaneous.

Miscellaneous Exercises on Book II.

1. Let ABCD be a quadrilateral, AC, BD its diagonals, and EF, GH lines joining the middle points of BC, AD, AB, CD It is required to prove that $AC^2 + BD^2 = 2\,EF^2 + 2\,GH^2$.

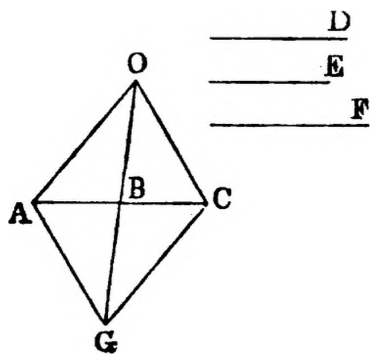
Dem.—Join GE, EH, HF, FG. Now GEHF is a parallelogram (I. xl., Ex. 6); $\therefore 2\,GH^2 + 2\,EF^2 = 2\,GE^2 + 2\,EH^2 + 2\,HF^2 + 2\,FG^2$ (x., Ex. 5) $= 4\,GE^2 + 4\,EH^2$.

Again, $GE = \frac{1}{2}\,AC$ (I. xl., Ex. 5), and $EH = \frac{1}{2}\,BD$; $\therefore 4\,GE^2 + 4\,EH^2 = AC^2 + BD^2$. Hence $2\,GH^2 + 2\,EF^2 = AC^2 + BD^2$.

2. Let AD, BE, CF be the medians.

Dem.—$AB^2 + AC^2 = 2\,BD^2 + 2\,AD^2$ (x., Ex. 2); $\therefore 2\,AB^2 + 2\,AC^2 = BC^2 + 4\,AD^2$; but $AO = \frac{2}{3}\,AD$; $\therefore AO^2 = \frac{4}{9}\,AD^2$; $\therefore 9\,AO^2 = 4\,AD^2$; hence $2\,AB^2 + 2\,AC^2 = BC^2 + 9\,AO^2$. Similarly $2\,AC^2 + 2\,CB^2 = AB^2 + 9\,CO^2$, and $2\,CB^2 + 2\,AB^2 = AC^2 + 9\,BO^2$; $\therefore 3\,(AB^2 + BC^2 + CA^2) = 9\,(AO^2 + BO^2 + CO^2)$. Hence $AB^2 + BC^2 + CA^2 = 3\,(AO^2 + BO^2 + CO^2)$.

3. **Sol.**—Construct the △ OCG, having OC = D, OG = 2 E, and CG = F. Bisect OG in B. Join CB, and produce it to A. Cut off AB = BC. Join AO. OA, OB, OC are the required lines.



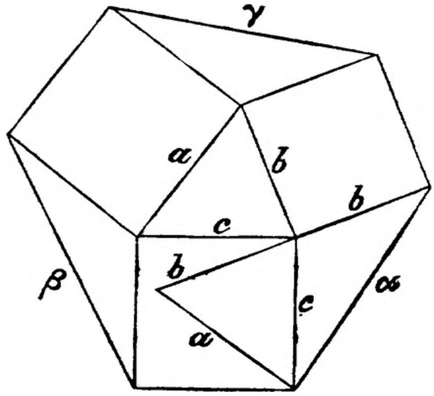
Dem.—The △s ABO, CBG are equal in every respect (I. iv.); $\therefore$ AO = CG = F, and OC = D, and OB = E.

4. Let ABCD be a quadrilateral; AC, BD its diagonals. Bisect AB, CD in EF. Join EF. It is required to prove that $AD^2 + BC^2 + AC^2 + BD^2 = AB^2 + DC^2 + 4\,EF^2$.

Dem.—Join CE, DE. Now $AD^2 + BD^2 = 2\,AE^2 + 2\,ED^2$ (x., Ex. 2), and $AC^2 + BC^2 = 2\,BE^2 + 2\,CE^2$; $\therefore AD^2 + BD^2 + AC^2 + BC^2 = 2\,AE^2 + 2\,BE^2 + 2\,CE^2 + 2\,DE^2$; but $2\,AE^2 + 2\,BE^2 = 4\,AE^2 = AB^2$; and $2\,CE^2 + 2\,DE^2 = 4\,DF^2 + 4\,EF^2 = DC^2 + 4\,EF^2$. Therefore $AD^2 + BD^2 + BC^2 + AC^2 = AB^2 + DC^2 + 4\,EF^2$.

5. Let *a*, *b*, *c* be the sides of the triangle. On *a*, *b*, *c* describe squares. Join the adjacent corners, and let the joining lines be

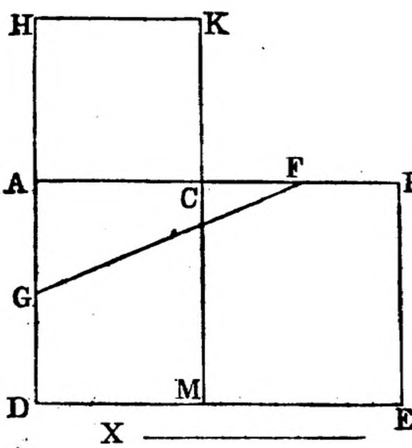
denoted by α, β, γ. It is required to prove that $\alpha^2 + \beta^2 + \gamma^2 = 3(a^2 + b^2 + c^2)$.



Dem.—Complete the construction, as in I. XLVII., Ex. 6. Now we have (X., Ex. 2) $\alpha^2 + a^2 = 2b^2 + 2c^2$; $\beta^2 + b^2 = 2c^2 + 2a^2$; and $\gamma^2 + c^2 = 2a^2 + 2b^2$. Add together, and we get $\alpha^2 + \beta^2 + \gamma^2 + (a^2 + b^2 + c^2) = 4(a^2 + b^2 + c^2)$; and $\therefore \alpha^2 + \beta^2 + \gamma^2 = 3(a^2 + b^2 + c^2)$.

6. Let AB be a given line. It is required to divide it into two parts at C, so that the rectangle contained by another given line X, and one segment BC, will be equal to AC².

Sol.—Erect AD ⊥ to AB, and equal to X. Complete the rectangular ▭ ABDE. Construct a square equal to ABDE, and let



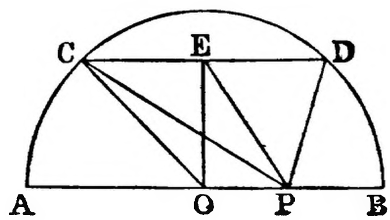
AF be one of its sides. Bisect AD in G. Join GF. Produce DA to H. Cut off GH = GF. In AB take AC = AH. C is the required point.

Dem.—Complete the square AHKC. Produce KC to meet DE

in M. Now DH . HA + AG^2 = GH^2 (VI.); but GH^2 = GF^2 = $AG^2 + AF^2$; ∴ DH . HA = AF^2; but AF^2 = ABDE (const.); ∴ the figure HM = BD. Reject DC, and HC = BM; but BM is the rectangle BC . BE; that is, BC . X; and HC is AC^2; ∴ BC . X = AC^2.

If we put $\frac{AB}{m} = X$, where m is any quantity, we get AB . BC = $m\, AC^2$.

7. **Dem.**—Bisect AB in O. Erect OE ⊥ to AB, and join OC, EP. Now (III., 3) CD is bisected at E; ∴ (x., Ex. 2)



$CP^2 + PD^2 = 2\,CE^2 + 2\,EP^2 = 2\,CE^2 + 2\,EO^2 + 2\,OP^2 = 2\,CO^2 + 2\,OP^2 = 2\,AO^2 + 2\,OP^2 = AP^2 + PB^2$ (IX.).

8. See "Sequel to Euclid," Book II., Prop. VII.

9. Let ABCDE be the pentagon; AC, BD, CE, AD, BE its diagonals. Bisect the diagonals. Let α be the line joining the middle points of AC, BD; β of BD, CE; γ of CE, AD; δ of AD, BE; and ϵ of BE, AC. It is required to prove that $3\,(AB^2 + BC^2 + CD^2 + DE^2 + EA^2) = AC^2 + BD^2 + CE^2 + AD^2 + BE^2 + 4\,(\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2)$.

Dem.—

$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4\,\alpha^2$ (XIII., Ex. 2).

$BC^2 + CD^2 + DE^2 + EB^2 = BD^2 + CE^2 + 4\,\beta^2$;

$CD^2 + DE^2 + EA^2 + AC^2 = CE^2 + DA^2 + 4\,\gamma^2$;

$DE^2 + EA^2 + AB^2 + BD^2 = DA^2 + EB^2 + 4\,\delta^2$;

$EA^2 + AB^2 + BC^2 + CE^2 = EB^2 + AC^2 + 4\,\epsilon^2$.

Add together, and we have

$$3\,(AB^2 + BC^2 + CD^2 + DE^2 + EA^2) = AC^2 + BD^2 + CE^2 + AD^2 + BE^2 + 4\,(\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2).$$

10. See "Sequel to Euclid," Book II., Prop. V.

11. See "Sequel to Euclid," Book II., Prop. VIII.

12. See "Sequel to Euclid," Book II., Prop. IX.

13. See "Sequel to Euclid," Book II., Prop. IX., Cor.

14. (1) **Dem.**—It is proved in Ex. 12 that

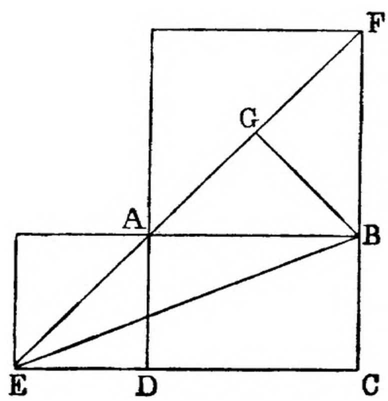
$$m\,AC^2 + n\,BC^2 = m\,AD^2 + n\,DB^2 + (m+n)\,DC^2;$$

but $m\,AC^2 + n\,BC^2$ is given (hyp.); $\therefore$ $m\,AD^2 + n\,DB^2 + (m+n)\,DC^2$ is given, and $m\,AD^2 + n\,DB^2$ is given; $\therefore$ $(m+n)\,DC^2$ is given; but $(m+n)$ is given; $\therefore$ DC^2 is given; $\therefore$ DC is given, and D is a given point. Hence the locus of the vertex is a ⊙, having D as centre, and DC as radius.

(2) This case can be proved in a similar manner by using Ex. 13.

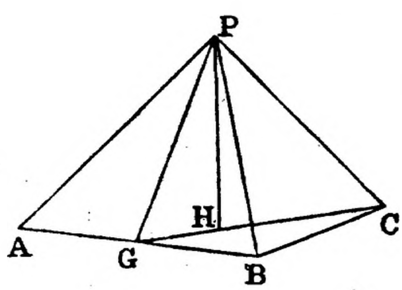
15. Let ABCD be a rectangle, of which AB, AD are adjacent sides. On AB, AD describe squares AF, AE. Draw the diagonals AF, AE. It is required to prove that AF . AE is equal to twice the rectangle AC.

Dem.—Let fall a ⊥ BG on AF. Now, because the ∠ ABF is right, $AF^2 = AB^2 + BF^2 = 2\,AB^2$. For a similar reason AE^2



$= 2\,AD^2$; hence $AF^2 . AE^2 = 4\,AB^2 . AD^2$; therefore $AF . AE = 2\,AB . AD$; that is, AF . AE is equal to twice the rectangle AC.

16. **Dem.**—Join AB, BC. Bisect AB in G. Join PG, CG,

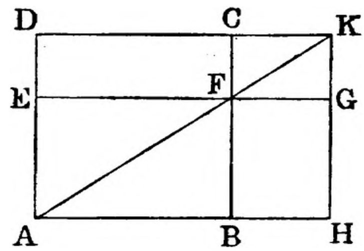


AP, BP, CP. Divide GC in H, so that HC = 2 GH. Join PH. Now $AP^2 + BP^2 = 2\,AG^2 + 2\,GP^2$ (x., Ex. 2), and $2\,PG^2 + PC^2$

$= 2\,GH^2 + HC^2 + 3\,HP^2$ (Ex. 12); $\therefore AP^2 + BP^2 + CP^2 = 2\,AG^2 + 2\,GH^2 + HC^2 + 3\,HP^2$; but $AP^2 + BP^2 + CP^2$ is given (hyp.); $\therefore 2\,AG^2 + 2\,GH^2 + HC^2 + 3\,HP^2$ is given; but $2\,AG^2$ is given, and $2\,GH^2$, and HC^2; hence $3\,HP^2$ is given; $\therefore$ HP is given, and the point H is given. Hence the locus of P is a circle.

17. Let ABCD be a square, and AEGH a rectangle of equal area. It is required to prove that the perimeter of ABCD is less than that of AEGH.

Dem.—ABCD = AEGH (hyp.). Take away the common part AEFB, and we have EDCF = BFGH; hence these must be the complements about the diagonal of a parallelogram; $\therefore$ if DC, AF, HG be produced, they are concurrent. Let them meet in K. Now DK is greater than DA; $\therefore$ the $\angle$ DAK is greater than DKA;



that is, CFK is greater than CKF; $\therefore$ CK is greater than CF, and therefore greater than DE. To each add CD + EA, and we get KD + EA; that is, GE + EA, greater than CD + DA. Hence the perimeter of the rectangle is greater than that of the square.

18. Let the transversal be divided by the lines, so that $m.AC = n.CB$; then $\dfrac{m}{n} = \dfrac{BC}{AC}$.

Dem.—$m.AD^2 + n.DB^2 = m.AC^2 + n.BC^2 + (m + n)\,CD^2$ (Ex. 12);

$$\therefore \frac{m}{n}AD^2 + DB^2 = \frac{m}{n}AC^2 + BC^2 + \left(\frac{m}{n} + 1\right)CD^2;\ \text{but}\ \frac{m}{n} = \frac{BC}{AC};$$

$$\therefore \frac{BC}{AC}.AD^2 + DB^2 = \frac{BC}{AC}.AC^2 + BC^2 + \left(\frac{BC}{AC} + 1\right)CD^2;$$

$$\therefore BC.AD^2 + AC.DB^2 = BC.AC^2 + AC.BC^2 + AB.CD^2;$$

$$\therefore BC.AD^2 + AC.DB^2 - AB.CD^2 = AC.CB\,(AC + CB);$$

$$\therefore BC.AD^2 + AC.DB^2 - AB.CD^2 = AB.BC.CA.$$

Lemma.—If a circle be described about an equilateral triangle, the square of the side of the triangle is equal to three times the square of the radius.

Dem.—Let BC be the side of the equilateral $\triangle$ ABC, and O the centre of the circumscribing circle. Join BO, and produce it to meet the circumference in D. Join DC, OC, OA.

Now, in the $\triangle^s$ AOB, BOC, the $\angle$ ABO = CBO (I. VIII.); $\therefore$ CBO is half an $\angle$ of an equilateral $\triangle$. In like manner BCO is half an $\angle$ of an equilateral $\triangle$, and $\therefore$ DOC is an $\angle$ of an equilateral $\triangle$, and OD = OC, being radii of the circle; $\therefore$ ODC is an equilateral $\triangle$; $\therefore$ DC = OC; but it has been shown that BCO is half an $\angle$ of an equilateral $\triangle$, and DCO an $\angle$ of an equilateral $\triangle$; $\therefore$ BCD is a right $\angle$; $\therefore BD^2 = BC^2 + CD^2 = BC^2 + CO^2$. Let BO be denoted by r, then $BD^2 = 4r^2$, and $OC^2 = r^2$; $\therefore 4r^2 = BC^2 + r^2$. And therefore $BC^2 = 3r^2$.

19. **Dem.**—Join AD, CD, CD′. Now in the $\triangle$ DCD′, $DD'^2 = DC^2 + CD'^2 + DC \,.\, CD'$ (XII., Ex. 1); $\therefore 6\,DD'^2 = 6\,DC^2 + 6\,CD'^2 + 6\,DC \,.\, CD'$.

Again, $AC^2 = 3\,CD^2$ (Lemma), and $CB^2 = 3\,CD'^2$; $\therefore AC^2 \,.\, CB^2 = 9\,CD^2 \,.\, CD'^2$; $\therefore AC \,.\, CB = 3\,CD \,.\, CD'$; $\therefore 2\,AC \,.\, CB = 6\,CD \,.\, CD'$; hence we have $6\,DD'^2 = 2\,AC^2 + 2\,CB^2 + 2\,AC \,.\, CB = AC^2 + CB^2 + (AC^2 + CB^2 + 2\,AC \,.\, CB) = AC^2 + CB^2 + AB^2$.

20.—**Dem.**—Let c be the hypotenuse; then $ab = cp$ (I., Cor. 1); $\therefore a^2 b^2 = c^2 p^2$; $\therefore a^2 b^2 = (a^2 + b^2)\, p^2 = a^2 p^2 + b^2 p^2$. Divide by $a^2 b^2 p^2$, and $\frac{1}{p^2} = \frac{1}{b^2} + \frac{1}{a^2}$.

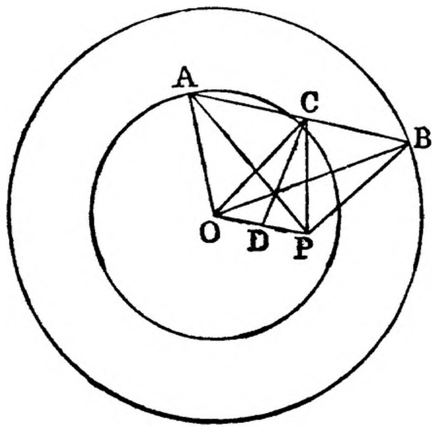
21. **Dem.**—Since ABD is an isoceles $\triangle$, $DC^2 - DB^2 = AC \,.\, CB$ (VI., Ex. 6) $= AB^2$ (hyp.). Hence $DC^2 = DB^2 + AB^2 = 2\,AB^2$.

22. Let a variable line AB, whose extremities rest on the circumferences of two given concentric $\odot^s$, subtend a right $\angle$ at a fixed point P. It is required to prove that the locus of its middle point C is a $\odot$.

Dem.—Join OA, OB, OP. Bisect OP in D. Join CO, CD, CP.

Now $AO^2 + OB^2 = 2\,BC^2 + 2\,CO^2$ (X., Ex. 2); but AO, OB are given, being radii of the given $\odot^s$; $\therefore 2\,BC^2 + 2\,CO^2$ is given; $\therefore BC^2 + CO^2$ is given; but BC = CP (I. XII., Ex. 2); $\therefore CO^2 + CP^2$ is given; that is, $2\,OD^2 + 2\,DC^2$ is given; but $2\,OD^2$ is

given, since OP is bisected in D ; ∴ $2\,DC^2$ is given ; ∴ DC is a



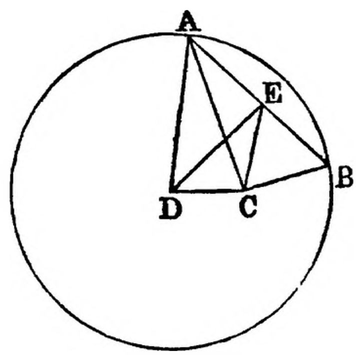
given line, and D is a fixed point. Hence the locus of C is a ⊙, having D as centre, and DC as radius.

BOOK III.

PROPOSITION III.

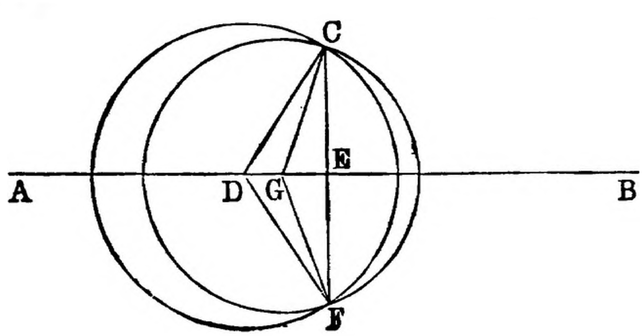
1. Let AB be the chord subtending a right ∠ at the point C. It is required to prove that the locus of the middle point of AB is a circle.

Dem.—Let D be the centre. Draw DE ⊥ to AB, and join CD, AD, CE.



Now (III.) AB is bisected in E; ∴ the lines AE, BE, CE are equal (I. XII., Ex. 2). Again, $AD^2 = AE^2 + ED^2 = ED^2 + EC^2$; but AD^2 is given, since AD is the radius; ∴ $ED^2 + EC^2$ is given, and the base DC is given; ∴ (II. X., Ex. 3), the locus of E is a circle.

2. Let AB be the given line, and C the given point. Take any



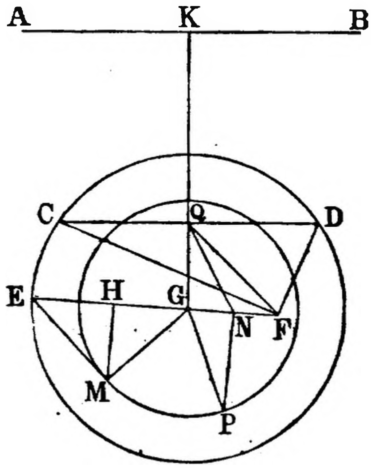
point D in AB. Join DC. With D as centre, and DC as radius,

describe a ⊙. From C let fall a ⊥ CE on AB, and produce it to meet the circumference in F. It is required to prove that every ⊙ having its centre in AB, and passing through C, must pass through F.

Dem.—Take any other point G in AB. Join GC. With G as centre, and GC as radius, describe a ⊙. Join FG. Now EC = EF (III.), and EG common, and the ∠ CEG = FEG; ∴ (I. IV.) CG = FG. Hence the second circle must pass through F.

3. Let CDE be the given circle, AB the given line, and F the given point. It is required to draw a chord in CDE which shall subtend a right ∠ at F, and be ∥ to AB.

Sol.—Let G be the centre of CDE. From G let fall a ⊥ GK on AB. Join FG, and produce it to meet the ⊙ in E. Bisect EG in H. Erect HM ⊥ to EG, and make it equal to GH.

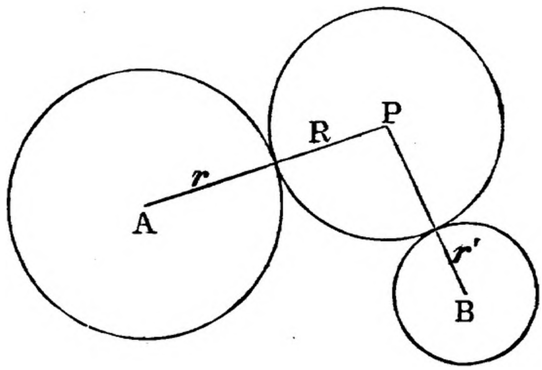


Join GM. Bisect FG in N, and erect NP ⊥ to FG. With G as centre, and GM as radius, describe a ⊙, meeting NP in P. With N as centre, and NP as radius, describe a ⊙, cutting GK in Q. Through Q draw CD ∥ to AB. CD is the required line.

Dem.—Join GP, GC, CF, QF, QN, FD. Now, since EG = 2 GH, $EG^2 = 4\,GH^2$; but $MG^2 = MH^2 + HG^2 = 2\,GH^2$. Hence $EG^2 = 2\,MG^2 = 2\,GP^2 = 2\,PN^2 + 2\,NG^2 = 2\,GN^2 + 2\,NQ^2$; but $2\,GN^2 + 2\,NQ^2 = QG^2 + QF^2$ (II. x., Ex. 2), and $EG^2 = GC^2$; ∴ $GC^2 = QG^2 + QF^2$; but $GC^2 = QC^2 + QG^2$; ∴ $QF^2 = QC^2$, and QF = QC; but QC = QD (III.); hence the three lines QC, QF, QD are equal; ∴ (I. XII., Ex. 2) the ∠ CFD is right.

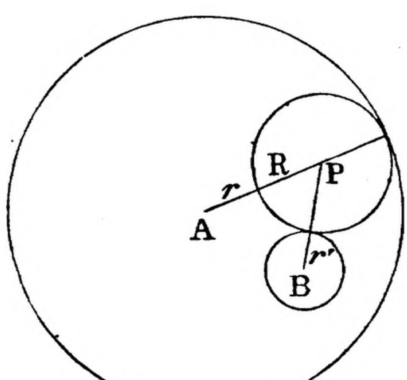
PROPOSITION XIII.

1. (1) **Dem.**—Let A, B be the centres of the fixed circles, and P the centre of the variable one. Join AP, BP; and let the radii be denoted by R, r, r'. Now $AP = R + r$, and $BP = R + r'$; $\therefore AP - BP = r - r'$.



(2) If the contact of the variable $\odot$ with the $\odot$ whose centre is B be of the second species, we have $AP = R + r$, and $BP = R - r'$; $\therefore AP - BP = r + r'$.

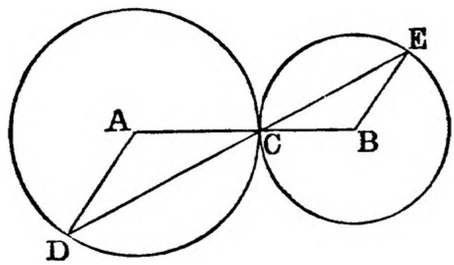
2. (1) **Dem.**—Let the $\odot$ whose centre is P touch that whose centre is A internally, and be touched by the one whose centre is B externally; then, denoting the radii as in the last Exercise, we get $AP = r - R$, $BP = r' + R$; and $\therefore AP + BP = r + r'$.



(2) If the $\odot$ whose centre is B touches the variable $\odot$ internally, we get $AP = r - R$, and $BP = R - r$; $\therefore AP + BP = r - r'$.

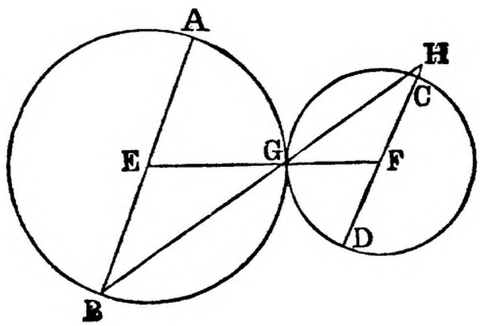
3. **Dem.**—Let A, B be the centres, and C the point of con-

tact. Join AB. Through C draw DE, meeting the ⊙s in D, E. Join AD, BE.



Now the ∠ ADC = ACD, and BCE = BEC; but ACD = BCE (I. xv.); ∴ ADC = BEC; and hence (I. xxvii.) AD is parallel to BE.

4. Let AB, CD be the diameters, G the point of contact, and E, F the centres. Join BG. It is required to prove that BG produced must pass through C.



Dem.—If possible, let it pass through H. Produce DC to meet BH. Join GE, GF.

Now the ∠ EBG = FHG (I. xxix.); but EBG = EGB = FGH; ∴ FHG = FGH; ∴ FG = FH; but FG = FC; ∴ FC = FH, which is absurd. Hence BG produced must pass through C. In like manner DG produced must pass through A.

PROPOSITION XIV.

1. (1) **Dem.**—Let ABC be the fixed circle, and AB the chord. From the centre D let fall a ⊥ DE on AB. Join AD.

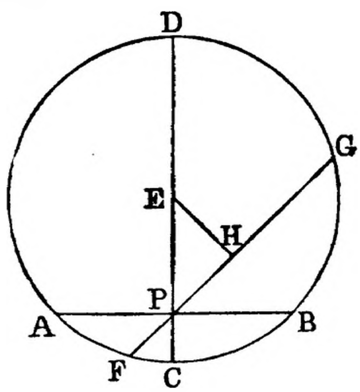
Now AB is bisected in E (iii.); ∴ AE is a line of given length, and AD is given, since it is the radius; but $AD^2 = AE^2 + DE^2$; ∴ DE is given, and the point D is given. Hence the locus of E is a circle.

(2) Let ABC be the ⊙, AB the chord, and E any fixed point in AB.

Dem.—Let D be the centre. Join AD, BD, ED. Now, because AB is given, and E is a fixed point in it, ∴ AE and EB are each given; ∴ AE . EB is given; and because ADB is an isosceles △, AE . EB = $BD^2 - DE^2$ (II. VI., Ex. 6); but AE . EB is given, and BD^2 is given, since BD is the radius; ∴ DE is given, and the point D is given. Hence the locus of E is a circle.

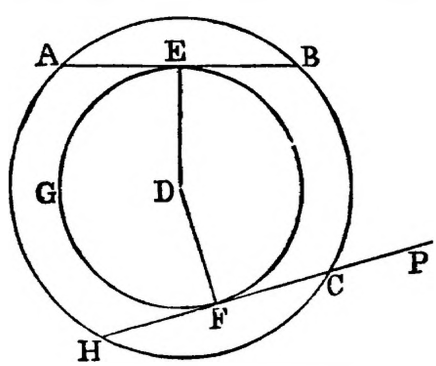
PROPOSITION XV.

1. Let ABC be the ⊙, and P the point. Through P draw a chord AB ⊥ to the diameter CD. It is required to prove that AB is the minimum chord.



Dem.—Through P draw any other chord FG; and from E, the centre, let fall a ⊥ EH on it. Now the ∠ EHP is right, ∴ EPH is acute; ∴ EP is greater than EH; ∴ (XV.) FG is greater than AB.

2. Let ABC be the given circle, AB the given chord, and P



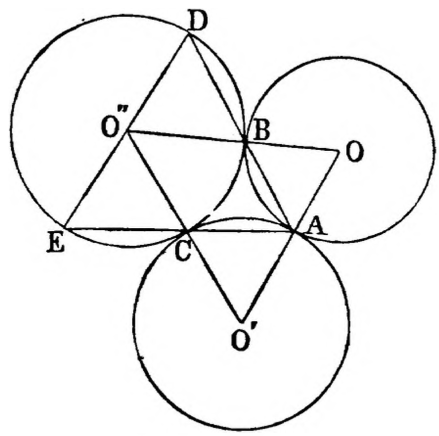
the point. It is required, through the point P, to draw a chord equal in length to AB.

Sol.—From the centre D let fall a ⊥ DE on AB. With D as centre, and DE as radius, describe a ⊙ EFG. Through P draw PCFH, touching EFG in F, and cutting ABC in C and H. CH is the chord required.

Dem.—Join DF. Now because DF = DE, ∴ (xiv.) CH = AB.

3. See "Sequel to Euclid," Book III., Prop. xv.

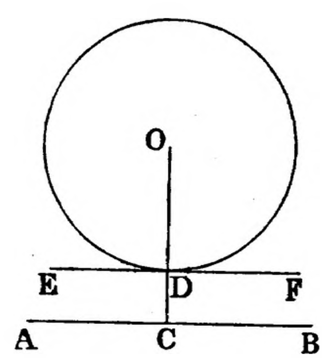
4. **Dem.**—Let O, O′, O″ be the centres. Now the lines joining OO′, O′O″, O″O must pass through A, C, B (xii.).



And because OA = OB, the ∠ OBA = OAB. Similarly, the ∠ O″BD = O″DB; but O″BD = OBA; hence O″DB = OAB, and ∴ O″D is parallel to OA. In like manner O″E is parallel to O′A; and hence O″D, O″E are in the same straight line.

PROPOSITION XVI.

1. **Dem.**—Let D be the common centre, and AB, CH the



chords of the greater which touch the less; then AB = CH (xiv.). See diagram to Prop. xv., Ex. 2.

2. Let AB be the given line, and O the centre of the given circle. It is required to draw a parallel to AB which shall touch the circle. (See last diagram.)

Sol.—Let fall a ⊥ OC on AB; and through D, where OC cuts the ⊙, draw EF parallel to AB. EF is the required line.

Dem.—Now the ∠ ODF = OCB (I. xxix.); ∴ ODF is a right ∠; hence (xvi.) EF touches the circle.

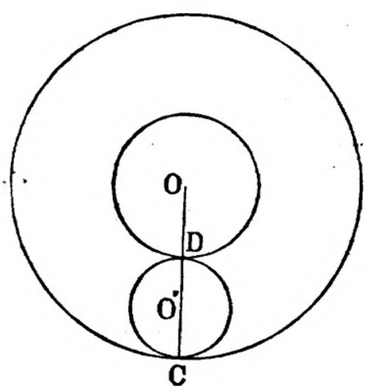
3. Let AB be the given line, and O the centre of the given circle. It is required to draw a perpendicular to AB which shall touch the circle.

Sol.—From O let fall a ⊥ OC on AB. Draw OF ∥ to AB, and from F, where it meets the ⊙, draw FB ∥ to OC. FB is the required line.

Dem.—The ∠s OCB, FBC are together equal to two right ∠s (I. xxix.); ∴ the ∠ FBC is right, and FB is ⊥ to AB.

4. (1) **Sol.**—Let O be the given point, and AB the given line. Let fall a ⊥ OC on AB. With O as centre, and OC as radius, describe a circle.

(2) Let O be the given point, and O′ the centre of the given circle. It is required to describe a ⊙ having its centre at O, and touching the ⊙ whose centre is O′.

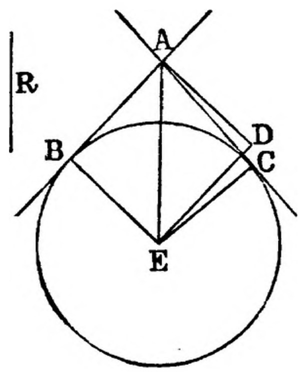


Sol.—Join OO′, and produce to meet the circumference of O′ in C; with O as centre, and OC as radius, describe a ⊙; or, with O as centre, and OD as radius, describe a ⊙. Hence there are two solutions.

5. Let AB, AC be the given lines, and R the given radius. It is required to describe a ⊙, touching AB, AC, and having a radius equal to R.

Sol.—Erect AD ⊥ to AB, and equal to R. Draw DE ∥ to AB.

Bisect the ∠ BAC by AE, meeting the line DE in E. E is the centre of the required ⊙.



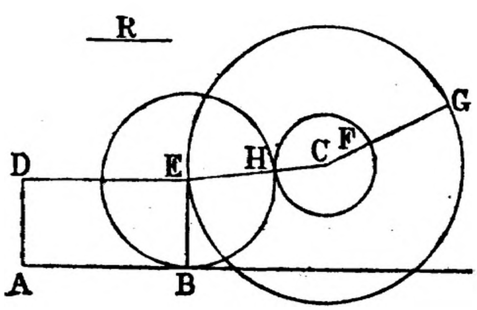
Dem.—Draw EB, EC, ⊥s to AB, AC.

Now the ∠ BAE = CAE, and the right ∠s ABE, ACE are equal, and AE common; ∴ (I. xxvi.) BE = CE; and the ⊙, with E as centre and BE as radius, will pass through C. If we produce BA, we can describe another ⊙ touching AC and the production of BA.

6. Let AB, AC be the given lines, and E the centre of one of the ⊙s which touch AB, AC.

Sol.—Join AE, and produce it. Join E to the points B, C, where the ⊙ touches AB, AC. Now, since the ∠s at B, C are right (xvi.), $AE^2 = AB^2 + BE^2 = AC^2 + CE^2$; but $BE^2 = CE^2$; ∴ $AB^2 = AC^2$; ∴ AB = AC, AE common, and the base BE = CE; ∴ (I. viii.) the ∠ BAE = CAE, ∴ the ∠ between the lines is bisected by the line joining their intersection to the centre of one of the circles. Hence the locus of the centres is the right line bisecting the angle between the two given lines.

7. (1) Let C be the centre of the given circle, AB the given



line, and R the radius. It is required to describe a ⊙ that shall

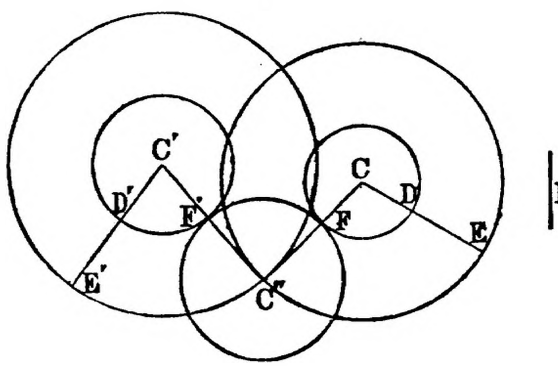
touch the ⊙ whose centre is C and the line AB, and have a radius equal to R.

Sol.—Take any point A in AB, and erect AD $\perp$ to it and $=$ R; draw DE $\parallel$ to AB; from C draw any radius CF, and produce it to G, so that FG $=$ R. With C as centre, and CG as radius, describe a ⊙ cutting DE in E. E is the centre of the required circle.

Dem.—Join CE, and draw EB $\parallel$ to AD. Now CG $=$ CE, and CF $=$ CH; $\therefore$ FG $=$ EH; but FG $=$ R; $\therefore$ EH $=$ R, and EB $=$ AD $=$ R, $\therefore$ EH $=$ EB, and the ⊙, with E as centre and EB as radius, will pass through H. Hence it will touch the given ⊙, the given line, and have a radius of given length.

(2) Let C, C′ be the centres of the given ⊙s, and R the given radius.

Sol.—Draw any two radii CD, C′D′, and produce them to E, E′, so that DE, D′E′ are each equal to R; with C, C′ as centres, and CE, C′E′ as radii, describe two ⊙s. Let them intersect in C″. C″ is the centre of the required circle.



Dem.—Join CC″, C′C″. Now CE $=$ CC″, and CD $=$ CF; hence DE $=$ FC″; but DE $=$ R (const.); $\therefore$ FC″ $=$ R. In like manner F′C″ $=$ R; $\therefore$ the ⊙ described with C″ as centre, and C″F as radius, will pass through F′, and touch the two ⊙s, and have the given radius.

PROPOSITION XVII.

2. Let O be the common centre. From any points A, B, on the outer circle tangents AC, BD are drawn to the inner one. It is required to prove that AC $=$ BD.

Dem.—Join OA, OB, OC, OD. Now (xvi.) the $\angle$s at C, D are right; $\therefore$ $OA^2 = OC^2 + CA^2$, and $OB^2 = OD^2 + DB^2$; but $OA^2 = OB^2$, and $OC^2 = OD^2$, $\therefore$ $AC^2 = BD^2$; $\therefore$ $AC = BD$.

3. Let ABCD be the quadrilateral. It is required to prove that $AB + CD = AD + BC$.

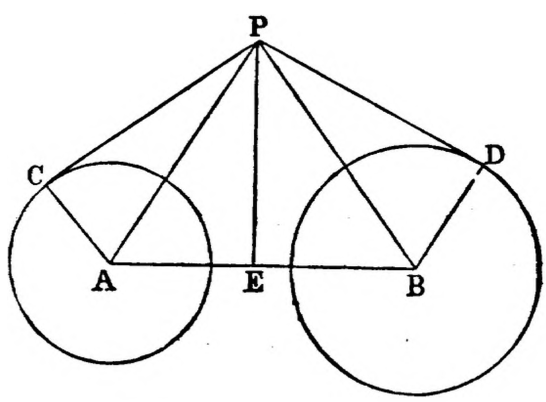
Dem.—Let E, F, G, H be the points of contact. Now (xvii., Ex. 1) $AE = AH$, and $BE = BF$; $\therefore$ $AB = AH + BF$. In like manner $CD = DH + CF$; $\therefore$ $AB + CD = AD + BC$.

4. **Dem.**—Let ABCD be the circumscribed parallelogram. Now $AB + CD = 2\ CD$, and $AD + BC = 2\ AD$; but $AB + CD = AD + BC$; $\therefore$ $2\ CD = 2\ AD$; $\therefore$ $CD = AD$. In like manner all the sides are equal. Hence ABCD is a lozenge.

Again, the line joining the centre to the intersection of tangents bisects the angle between the tangents; conversely, the line bisecting the angle between the tangents passes through the centre; therefore AC passes through the centre. Similarly, BD passes through the centre. Hence E is the centre.

5. **Dem.**—$OB = OD$, and OP common, and the base $BP = DP$; $\therefore$ (I. viii.) the $\angle$ $BOP = DOP$. Again, $OB = OD$, OF common, and the $\angle$ $BOF = DOF$; $\therefore$ (I. iv.) the $\angle$ $OFB = OFD$. Hence each is a right $\angle$, and OP is $\perp$ to BD.

6. Let A, B be the centres of the $\odot$s. Let P be a point from which the tangents PC, PD to the $\odot$s are equal. It is required to prove that the locus of P is a right line.

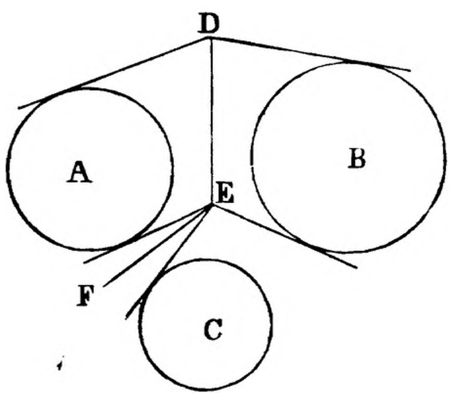


Dem.—Join AC, AP, BD, BP, and from P let fall a $\perp$ PE on AB. Now $AP^2 = AC^2 + CP^2$; $\therefore$ $CP^2 = AP^2 - AC^2$. In like manner $DP^2 = BP^2 - BD^2$; but $CP^2 = DP^2$; $\therefore$ $AP^2 - AC^2 = BP^2 - BD^2$; $\therefore$ $AP^2 - BP^2 = AC^2 - BD^2$; but $AC^2 - BD^2$ is given, since AC, BD are the radii of the $\odot$s; $\therefore$ $AP^2 - BP^2$ is given,

$\therefore AE^2 - EB^2$ is given; $\therefore$ E is a given point; hence EP is given in position, and therefore the locus of P is the right line EP (called the radical axis of the two circles).

Cor.—To construct the line EP, join the centres, divide the joining line in E, so that $AE^2 - EB^2 = AC^2 - BD^2$; and erect EP $\perp$ to AB.

7. Let the three circles be denoted by A, B, C. It is required



to find a point such that the tangents from it to A, B, C shall be equal.

Sol.—Find a line DE, such that the tangents from any point of it to A and B will be equal (XVII., Ex. 6); and find a line FE, such that the tangents from any point of it to A and C shall be equal. E, where the lines DE, FE intersect, is the required point.

8. **Dem.**—OBP is a right-angled Δ, and BF is $\perp$ to OP (XVII., Ex. 5); $\therefore$ (I. XLVII., Ex. 1) $OB^2 = OF \cdot OP$.

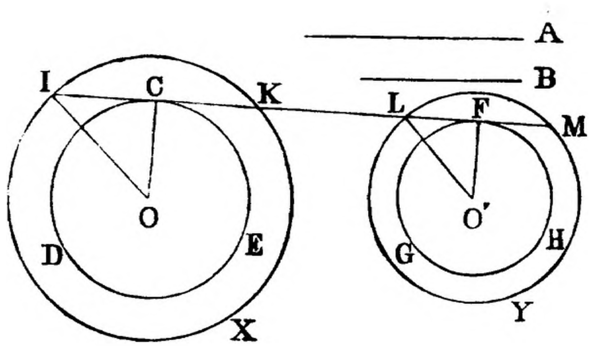
9. Let AB, AC be two fixed tangents, and EF a variable tangent, cutting AB, AC in E, F, and touching the $\odot$ in D. Let O be the centre. Join OE, OF. It is required to prove that the $\angle$ EOF is constant.

Dem.—Join OB, OC, OD. Now (I. VIII.) the $\angle$ EOD = EOB; $\therefore$ EOD = $\frac{1}{2}$ BOD. In like manner FOD = $\frac{1}{2}$ COD; $\therefore$ EOF = $\frac{1}{2}$ BOC; but the $\angle$ BOC is constant, since the tangents AB, AC are fixed; $\therefore$ the $\angle$ EOF is constant.

10. (1) See "Sequel to Euclid," Book III., Prop. VII.

(2) Draw a line cutting two circles, X, Y, so that the intercepted chords shall be of given lengths A, B.

Sol.—Let O, O′ be the centres of X, Y; R, R′ their radii. Then with O, O′ as centres, describe ⊙s CDE, FGH, the squares



of whose radii shall be equal to $R^2 - \frac{1}{4}A^2$, and $R'^2 - \frac{1}{4}B^2$ respectively, and draw the line IM a common tangent to both circles. IM is the line required.

Dem.—Let C, F be the points of contact. Join OC, OI; O′F, O′L. Now $OC^2 = OI^2 - IC^2 = R^2 - IC^2$; but $OC^2 = R^2 - \frac{1}{4}A^2$ (const.); ∴ $IC^2 = \frac{1}{4}A^2$. Hence $IC = \frac{1}{2}A$; but $IC = \frac{1}{2}IK$ (III. III.); ∴ IK = A. In like manner LM = B.

PROPOSITION XXI.

1. (1) Let ABC be a △, whose base BC, and vertical ∠ BAC, are given. From B, C let fall ⊥s BE, CF on AC, AB, and let them intersect in G. It is required to find the locus of G.

Dem.—The four ∠s A, F, G, E of the quadrilateral AFGE are together equal to four right ∠s (I. XXXII., Cor. 3); but the ∠s E, F are right; ∴ the ∠s A, G are together equal to two right ∠s; but A is given (hyp.); ∴ G is given; ∴ (I. XV.) the ∠ BGC is given. And hence (XXI., Cor. 2), the locus of G is a circle.

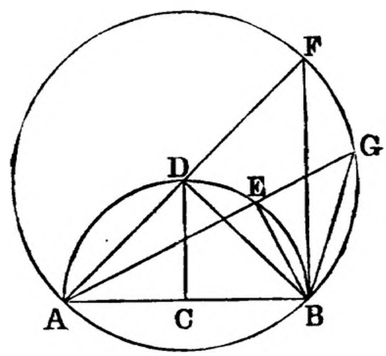
(2) Let the internal bisectors meet in D. Now, the three ∠s of the △ ABC are equal to two right ∠s; but the ∠ A is given; ∴ the sum of the ∠s B, C is given; ∴ half their sum is given;

that is, DBC + DCB is given; ∴ the ∠ BDC is given; and hence (xxi., Cor. 2) the locus of D is a circle.

(3) Let the external bisectors meet in D. Then, as before, the sum of the ∠s B, C is given; ∴ (I. xxxii., Ex. 14) the ∠ D is given. Hence (xxi., Cor. 2) the locus of D is a circle.

(4) **Dem.**—Let the external bisector of the ∠ C, and the internal bisector of B meet in D; then the ∠ BDC = $\frac{1}{2}$ BAC (I. xxxii., Ex. 2); ∴ the ∠ BDC is given. Hence (xxi., Cor. 2) the locus of D is a circle.

2. Let AB^2 be equal to the sum of the squares of the two lines.



It is required to prove that their sum is a maximum when the lines are equal.

Sol.—Upon AB describe a semicircle ADB. Bisect AB in C, and erect CD ⊥ to AB. Join AD, BD. In ADB take any other point E. Join AE, BE. Produce AD to F, so that DF = DB. Join BF. Produce AE to G, so that EG = EB, and join BG.

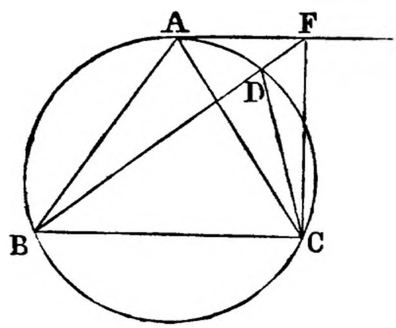
Dem.—The ∠ DFB = DBF (I. v.); but BDF is a right ∠; ∴ DFB is half a right ∠. Similarly, EGB is half a right ∠; hence (xxi., Cor. 1) the four points A, F, G, B are concyclic. Now, since D is a point in a ⊙ from which the three equal lines DA, DB, DF are drawn to the circumference, D is the centre; ∴ AF is the diameter; but the diameter is the greatest chord; ∴ AF is greater than AG; that is, the sum of AD and DB is greater than the sum of AE and EB.

3. Let there be two △s ADB, AEB on the same base AB, and having equal vertical ∠s, and let ADB be isosceles. It is required to prove that the sum of the sides AD and DB is greater than the sum of the sides AE and EB. (Diagram, Ex. 2.)

Dem.—Produce AD to F, so that DF = DB. Join BF. Produce AE to G, so that EG = EB, and join BG. Now the ∠ DFB = DBF (I. v.); but ADB = DFB + DBF (I. xxxii.); ∴ ADB = 2 DFB. Similarly, AEB = 2 EGB; but ADB = AEB (hyp.); ∴ DFB = EGB; and ∴ (xxi., Cor. 1) the points A, F, G, B are concyclic; and it can be shown, as in Exercise 2, that AD + DB is greater than AE + EB.

4. **Dem.**—Let ABC be an inscribed △. Then if any two sides AC, CB be unequal, by supposing the points A, B to remain fixed while C varies, the perimeter will be increased by making AC, CB equal. Hence, when the three sides AB, BC, CA become all equal, the perimeter will be a maximum.

Lemma.—Let ABC, DBC be two △s on the same base inscribed



in a circle, of which ABC is isosceles. It is required to prove that the area of ABC is greater than the area of BDC.

Dem.—Through A draw AF, touching the ⊙. Produce BD to meet it in F, and join CF. Now the ∠ FAC = ABC (xxxii.) = ACB; ∴ AF is ∥ to BC; hence (I. xxxvii.) the △ BFC = BAC; but BFC is greater than BDC; ∴ BAC is greater than BDC. Similarly it can be shown that BAC is greater than any other △ inscribed in the ⊙, having BC for base, whose sides are unequal. Hence the area of the isosceles △ is a maximum.

5. Let ABCDE be a polygon inscribed in a circle. It is required to prove that the area is a maximum when all the sides are equal.

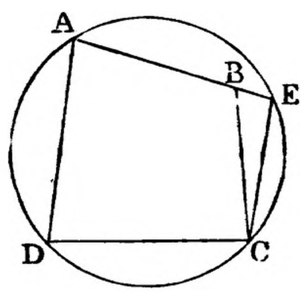
Dem.—Join AC. Now, if we suppose the point B to move about whilst the others remain fixed, when AB = BC, the △ ABC will be a maximum (Ex. 5), and therefore the area of the whole

figure will be increased. In like manner, if any other of the sides be unequal, we can increase the area by making them equal. Hence the area will be a maximum when all the sides are equal.

PROPOSITION XXII.

1. Let ABCD be a quadrilateral, whose opposite ∠s B, D are supplemental. It is required to prove that it is cyclic.

Dem.—If not, let the ⊙ through A, D, C, intersect the line



AB produced in E. Join CE. Now the ∠s ADC, CBA are together equal to two right ∠s (hyp.), and the ∠s ADC, CEA are equal to two right ∠s (xxii.) Reject ADC, and we have the ∠ CBA = CEA, which is impossible (I. xvi.). Hence the ⊙ must pass through B.

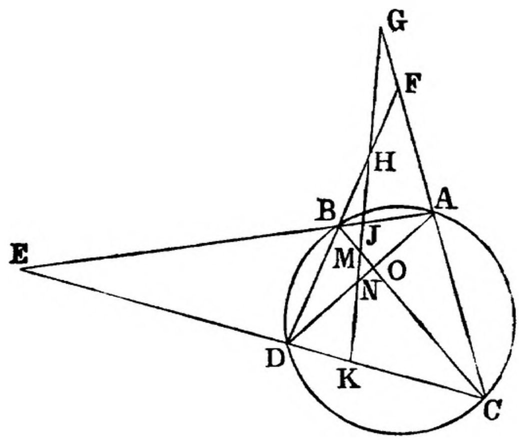
2. Let ABCDEF be a hexagon inscribed in a ⊙. It is required to prove that the sum of the alternate ∠s ABC, CDE, EFA is equal to four right angles.

Dem.—Join CF. Now the ∠s ABC, CFA are together equal to two right ∠s (xxii.), and the ∠s CDE, EFC are equal to two right ∠s. Hence, by addition, the ∠s ABC, CDE, EFA are equal to four right angles.

3. (1) Let ABDC be a cyclic quadrilateral, and let the opposite sides meet in E, F. Draw any line GK, cutting the four sides, and making the ∠ EJK = EKJ. It is required to prove that the ∠ GHF = HGF.

Dem.—The ∠s BDC and BAC are equal to two right ∠s (xxii.), and the ∠s BAC, BAG equal to two right ∠s. Reject the

∠ BAC, and we have the ∠ BDC = BAG, and the ∠ DKJ = AJG (hyp.); ∴ the remaining ∠ DHK = AGJ; that is, the ∠ GHF = HGF.



(2) Let GK cut the diagonals in M, N. It is required to prove the ∠ OMN = ONM.

Dem.—The ∠ EJK = EKJ (hyp.), and the ∠ ABC = ADC (XXI.); ∴ the remaining ∠ BMJ = DNK; that is, the ∠ OMN = ONM.

4. Bisect the ∠ AEC by ES, meeting the diagonals in Q, R. From O let fall a ⊥ OP on ES. It is required to prove that OP bisects the ∠ QOR.

Dem.—The ∠ ABC = BER + BRE (I. XXXII.), and ADC = DEQ + DQE; but (XXI.) ABC = ADC; ∴ BER + BRE = DEQ + DQE; but BER = DEQ (hyp.); ∴ BRE = DQE = OQR, and the ∠ OPR = OPQ. Hence the ∠ ROP = QOP.

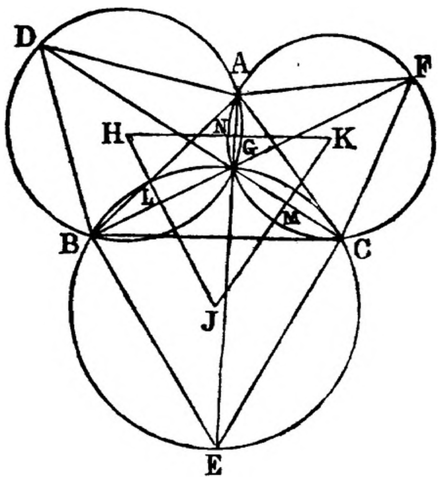
5. Let ABCDEF be a cyclic hexagon, having the side AB ∥ to DE, and BC to EF. It is required to prove that the side AF is ∥ to CD.

Dem.—Join CF. Now the ∠ ABC = DEF (I. XXIX., Ex. 8); and since ABCF is a cyclic quadrilateral, the ∠s ABC, AFC are together equal to two right ∠s. For the same reason the ∠s DCF, DEF are equal to two right ∠s; ∴ the ∠s ABC and AFC = DCF and DEF; but ABC = DEF; ∴ AFC = DCF. And hence (I. XXVII.) AF is ∥ to CD.

6. **Dem.**—Join AB. Now the ∠ BAD = BFD (XXI.), and BAC = BEC; ∴ BFD = BEC. And hence (I. XXVIII.) CE is ∥ to DF.

7. On the sides of any $\triangle$ ABC, equilateral $\triangle^s$ are described, BF and CD joined and intersecting in G. Join AG, EG. It is required to prove that AG and GE are in the same straight line.

Dem.—Since AB = AD, and AC = AF, and the $\angle$ BAD



= CAF; to each add BAC, therefore the $\angle$ DAC = BAF; hence (I. IV.) the $\angle$ ADC = ABF, and ACD = AFB. Now, because the $\angle$ ACG = AFG, AFCG is a cyclic quadrilateral; hence the $\angle^s$ AFC, AGC are together equal to two right $\angle^s$ (XXII.); similarly ADBG is a cyclic quadrilateral; and the $\angle^s$ ADB, AGB are equal to two right $\angle^s$; $\therefore$ these four $\angle^s$ are together equal to four right $\angle^s$, and the $\angle^s$ AGB, BGC, AGC are equal to four right $\angle^s$. Reject the $\angle^s$ AGB, AGC, and we have the $\angle$ BGC equal to the sum of AFC and ADB. To each add BEC, and we have BGC + BEC = AFC + ADB + BEC; but these three $\angle^s$ are equal to two right $\angle^s$, since each is an $\angle$ of an equilateral $\triangle$; $\therefore$ BGC, BEC are equal to two right $\angle^s$; and hence BGCE is a cyclic quadrilateral; $\therefore$ the $\angle$ EGC = EBC; $\therefore$ EGC is equal to an $\angle$ of an equilateral $\triangle$, and therefore equal to AFC; but AFC and AGC are equal to two right $\angle^s$; $\therefore$ EGC and AGC are equal to two right $\angle^s$, and hence (I. XIV.) AG and EG are in the same straight line. Therefore AE, BF, CD are concurrent.

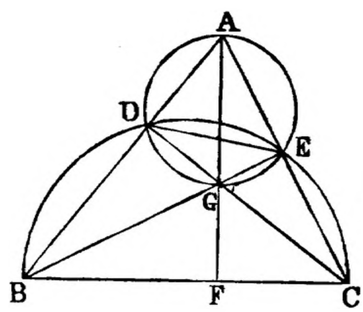
8. If we join the centres H, J, K, it is required to prove that HJK is an equilateral triangle.

Dem.—Let HJ, JK, HK cut BG, CG, AG in the points L, M, N. Now, because the ∠ˢ L, N are right (III., Cor. 4), ∴ the ∠ˢ H, G are equal to two right ∠ˢ, and the ∠ˢ G, D are equal to two right ∠ˢ; hence the ∠ H = D; ∴ H is an ∠ of an equilateral △. Similarly, K is an ∠ of an equilateral △. Hence the △ HJK is equilateral.

9. Let ABCD be the quadrilateral, O the centre of the inscribed circle, and E, F, G, H the points of contact. Join O to A, B, C, D. It is required to prove that the ∠ˢ AOB, DOC are supplemental.

Dem.—Join OE, OF, OG, OH. Now the ∠ AOB = half sum of the ∠ˢ EOH, EOF (XVII., Ex. 9), and the ∠ DOC = half sum of GOH, GOF; but the sum of EOH, EOF, GOH, GOF is four right ∠ˢ; ∴ AOB and DOC are together equal to two right angles.

10. Let ABC be a △, whose perpendiculars CD, BE intersect

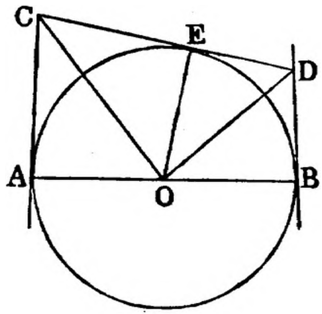


in G. Join AG, and produce it to meet BC in F. It is required to prove that AF is ⊥ to BC.

Dem.—Join DE. Now, because each of the ∠ˢ ADG, AEG is right, ADGE is a cyclic quadrilateral; hence the ∠ DEG = DAG (XXI.) Again, since the ∠ˢ BDC, BEC are right, the points B, D, E, C are concyclic, and therefore the ∠ DEB = DCB; ∴ DAG = DCB, and DGA = FGC (I. xv.); ∴ ADG = AFC; but ADG is a right ∠; ∴ AFC is a right ∠, and AF is perpendicular to BC.

11. Let a variable tangent CD meet two parallel tangents AC, BD. Join the centre O to C, D. It is required to prove that the ∠ DOC is right.

Dem.—Draw the diameter AB, and join O to the point E where CD touches the circle.



Now the ∠ DOC is equal to half the sum of the ∠s EOB, EOA (XVII., Ex. 9); but EOB and EOA are together equal to two right ∠s; ∴ the ∠ DOC is right.

12. See "Sequel to Euclid," Book III., Prop. XII.

13. Let ABCDEF be the hexagon, O the centre of the inscribed circle, and G, H, J, K, L, M the points of contact of the hexagon and circle. Join O to the points A, B, C, D, E, F. It is required to prove that the sum of the ∠s AOB, COD, EOF is two right angles.

Dem.—Join O to the points G, H, J, K, L, M. Now the ∠ AOB = $\frac{1}{2}$MOH (XVII., Ex. 9), COD = $\frac{1}{2}$HOK, and EOF = $\frac{1}{2}$ KOM; ∴ the sum of AOB, COD, EOF is two right ∠s.

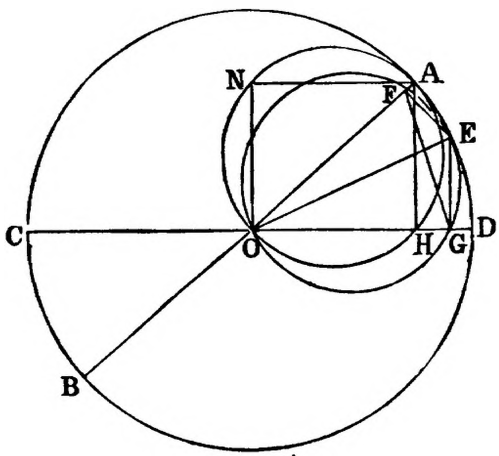
PROPOSITION XXVIII.

1. Let AB, CD be the two diameters given in position. Take any point E in the circumference, and let fall ⊥s EF, EG on AB, CD. Join FG. It is required to prove that FG is given in magnitude.

Dem.—Join OE, and from A let fall a ⊥ AH on CD. Now, since the ∠ OHA is right, the circle on OA as diameter will pass through H (XXXI.); and because the ∠s OFE, OGE are right, the ⊙ on OE as diameter will pass through F and G; but OA = OE; ∴ the ⊙s on OA and OE are equal, and the ∠ AOH is in both those ⊙s; ∴ the arc AH is equal to the arc FG (XXVI.), and therefore the chord AH = FG; but AH is given in magnitude, since it is a ⊥ from the extremity of one of the

diameters given in position on the other. Hence FG is given in magnitude.

2. Let OA, OD be two lines given in position, and FG a line



of given length sliding between them. At the extremities of FG, perpendiculars EF, EG are erected to OA, OD. It is required to prove that the locus of E, where these perpendiculars meet, is a circle. (Diagram to Ex. 1).

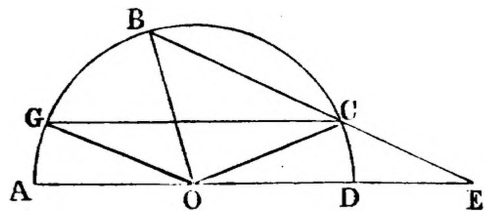
Dem.—Join OE. Erect ON ⊥ to OD, and equal to FG; draw NA ∥ to OD.

Now, because ONA is a right ∠, the ⊙ described on OA as diameter will pass through N; for a similar reason, the ⊙ on OE as diameter will pass through F and G. Now since ON and FG are equal, and subtend equal angles OAN, FOG in the ⊙s OAN, FOG, the ⊙s are equal; therefore the diameters OA, OE are equal. Again, since ON = FG, ON is given, and AN is ∥ to OD; ∴ the point A is given, and hence the line OA is given in magnitude; but OE = OA; ∴ OE is given in magnitude, and the point O is given. Hence the locus of E is a ⊙, having O as centre and OE as radius.

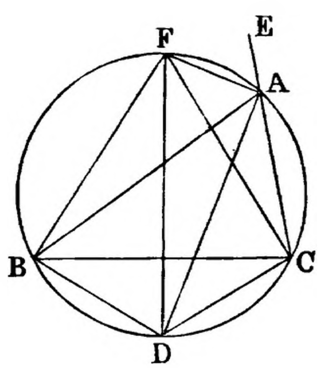
PROPOSITION XXX.

1. **Dem.**—Let O be the centre. Through C draw CG ∥ to DA. Join OB, OC, OG. Now the ∠ GCO = COE (I. xxix.); but GCO = CGO, and CGO = AOG, ∴ DOC = AOG; ∴ the arc

DC = AG (xxvi.). Again, the ∠ GOB is double GCB (xx.); but GCB = AEB (I. xxix.), and AEB = COE; because CE = OC (hyp.); ∴ GOB is double DOC; hence the arc GB is double CD, and therefore the arc AB is three times the arc CD.



2. (1) Let AD be the internal bisector of the vertical ∠ of the △ ABC. Join BD, CD. It is required to prove that BD = CD.



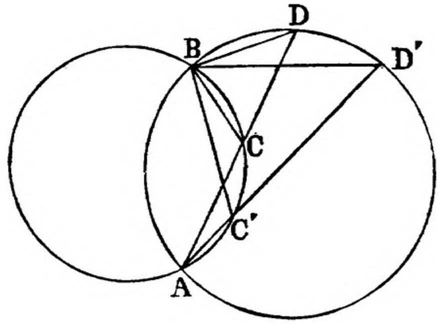
Dem.—Because the ∠ BAD = CAD, the arc BD = CD (xxvi.), and therefore the chord BD = CD (xxix.).

(2) Produce CA to E. Bisect the ∠ BAE by AF, meeting the circumference in F. It is required to prove that the point F is equally distant from B and C.

Dem.—Join BF, CF. Now the ∠s FBC and FAC are together equal to two right ∠s (xxii.), and FAC and FAE are equal to two right ∠s (I. xiii.); ∴ the ∠ FBC = FAE. Again, the ∠ BAF = BCF (xxi.); but BAF = FAE, BCF = FAE; ∴ BCF = FBC; ∴ BF = CF.

3. **Dem.**—The ∠ ADB = AD′B (xxi.), and the ∠ ACB = AC′B; but the ∠s ACB and DCB are together equal to two right ∠s,

and the $\angle$s AC′B, D′C′B are together equal to two right $\angle$s; $\therefore$ the $\angle$ DCB = D′C′B; and hence (I. XXXII., Cor. 2) the remaining $\angle$s DBC, D′BC′, are equal.

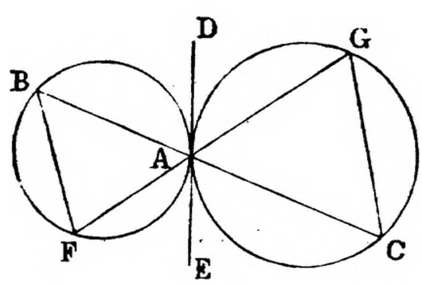


4. **Dem.**—Join AD, DB. Now because the $\angle$ ACD = BCD, the line AD = BD. Again, the $\angle$s DBC, DAC, together equal to two right $\angle$s (XXII.), and the $\angle$s DBC, DBF equal to two right $\angle$s (I. XIII.); $\therefore$ the $\angle$ DAE = DBF, and the right $\angle$s DEA, DFB are equal; $\therefore$ (I. XXVI.) AE = BF. Hence AC − CE = CF − CB; $\therefore$ AC + CB = CF + CE = 2 CE; because CF = CE.

PROPOSITION XXXII.

1. Let the circles touch in A. Through A draw any line BAC. It is required to prove that BAC divides the circles into similar segments.

Dem.—Through A draw a common tangent DE; take any



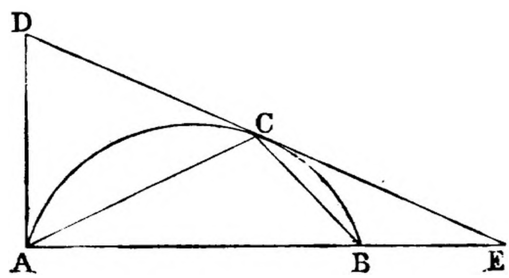
points F, G, in the $\odot$s. Join AF, BF, AG, CG. Now the $\angle$ BAD = AFB (XXXII.), and the $\angle$ CAE = AGC; but BAD = CAE (I. XV.);

∴ AFB = AGC; and hence the segments AFB, AGC are similar.

2. Let the circles touch in A. Through A draw two lines BC, FG, meeting the ⊙ˢ in B, C; F, G. Join BF, CG. It is required to prove that BF, CG are parallel.

Dem.—Through A draw a common tangent DE. Now it may be proved, as in Ex. 1, that the ∠ AFB = AGC; hence (I. xxvii.), BF is parallel to CG.

3. **Dem.**—Join AC, BC. Now the lines CA, CD, CE are



equal (I. xii., Ex. 2); ∴ the ∠ AEC = EAC; but (xxxii.) EAC = BCE; hence the ∠ BCE = BEC; ∴ BCE and BEC = 2 BEC; ∴ (I. xxxii.) the ∠ CBA = 2 BEC; but BEC = CAB, since CE = CA; ∴ CBA = 2 CAB. Hence the arc AC = 2 CB.

4. (1) See "Sequel to Euclid," Book III., Prop. iii.

(2) **Dem.**—Let GBF and ECH be the tangents to the circles at the points B, C. Join CF, CG. Now the ∠ GFC = FCH (I. xxix.); but FCH = FGC (xxxii.); ∴ GFC = FGC; and hence the chords GC, FC are equal.

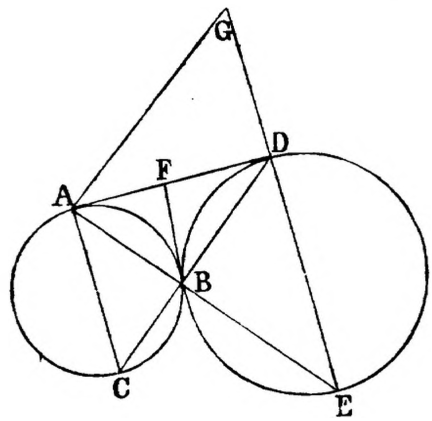
5. (1) Let the circles ABC, DBE touch at B. Draw a common tangent AD. Join AB, DB. It is required to prove that the angle ABD is right.

Dem.—Draw a common tangent BF. Now AF = BF (xvii., (Ex. 1); ∴ the ∠ ABF = BAF; and because BF = DF, the ∠ BDF = DBF; ∴ the ∠ ABD = BAD + BDA, and hence (I. xxxii., Cor. 7) the ∠ ABD is right.

(2) **Dem.**—Produce AB, DB to meet the circumferences in E, C. Join AC, DE. Produce ED to G, and draw AG parallel to CD.

Now, because the ∠ ABD is right, EBD is right, and therefore ED is a diameter, and hence (xix.) the ∠ ADE is right; ∴ AD

is ⊥ to EG. Again, since AG, CD are parallel, the ∠ GAE

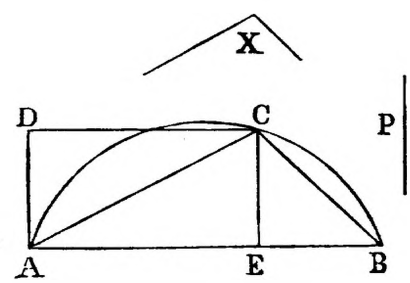


= DBE, and is therefore a right ∠. Hence (I.] XLVII., Ex. 2) $AD^2 = DE \cdot DG = DE \cdot AC$.

PROPOSITION XXXIII.

1. (1) Let AB be the base, X the vertical ∠, and P the perpendicular.

Sol.—On AB describe a segment ACB containing an ∠ equal to X (XXXIII.). Erect AD ⊥ to AB, and = P. Through D draw



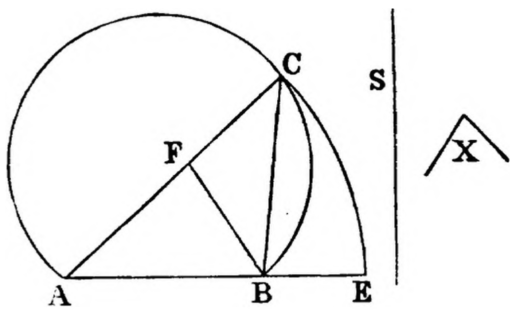
DC ∥ to AB, cutting the circle in C. Join AC, CB. ACB is the required triangle.

Dem.—Let fall a ⊥ CE on AB. The vertical ∠ ACB = X; AB is the base, and the ⊥ CE = AD = P.

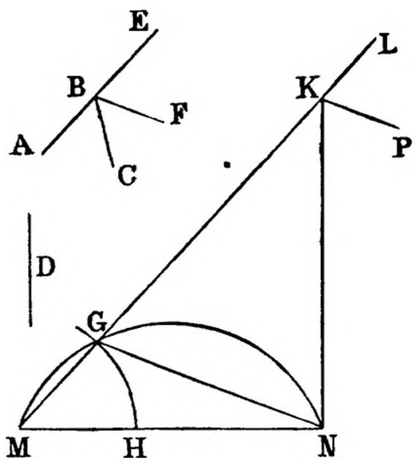
(2) Let the sum of the sides be equal to S.

Sol.—On AB describe a segment ACB containing an ∠ equal $\frac{1}{2}$ X. Produce AB to E. Cut off AE = S. With A as centre, and AE as radius, describe a ⊙, cutting ACB in C. Join AC, BC, and at the point B in the line BC make the ∠ FBC = FCB. AFB is the required triangle.

Dem.—FC = FB (I. VI.); ∴ AC = AF + FB; but AC = AE = S; ∴ AF + FB = S; and the ∠ AFB = FBC + FCB (I. XXXII.) = 2 FCB = X.



(2) Let MN be the base, D the difference of sides, and ABC the vertical angle.

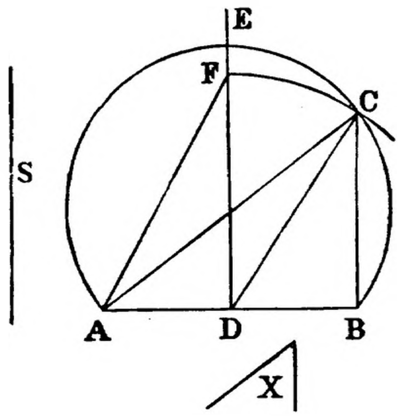


Sol.—Produce AB to E. Bisect the ∠ CBE by BF. On MN describe a segment MGN containing an ∠ = ABF; in MN take MH = D. With M as centre, and MH as radius, describe a ⊙, cutting MGN in G. Join MG, NG. Produce MG, and at the point N in GN make the ∠ GNK = NGK. MKN is the required triangle.

Dem.—Produce MK to L, and draw KP parallel to GN. Now KN = KG (I. VI.); ∴ MG is the difference between MK and NK; but MG = MH = D; ∴ the difference between MK and NK is equal to D. Again, the ∠ PKN = GNK (I. XXIX.), and LKP = KGN; but GNK and KGN are equal (const.); ∴ PKN and LKP are equal; and since the ∠ MKP = MGN = ABF,

the $\angle$ LKF = EBF ; but LKP = NKP, and EBF = FBC, $\therefore$ FBC = NKP. Hence the $\angle$ MKN = ABC.

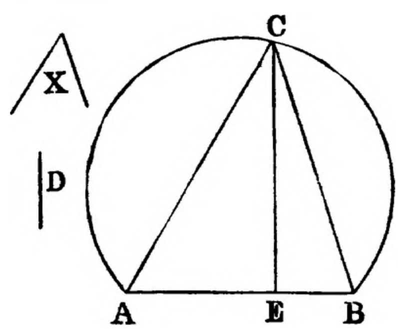
(3) Let AB be the given base, X the vertical $\angle$, and let the sum of the squares of the sides be equal to 2 S^2.



Sol.—On AB describe a segment containing an $\angle$ = X. Bisect AB in D, and erect DE $\perp$ to AB ; from A inflect AF on DE = S (I. II., Ex. 2). With D as centre, and DF as radius, describe a $\odot$, cutting ACB in C. Join AC, BC. ACB is the Δ required.

Dem.—Join CD. Now, DF = DC ; $\therefore$ $DF^2 = DC^2$; $\therefore$ $AD^2 + DF^2 = AD^2 + DC^2$; $\therefore$ AF^2, that is $S^2 = AD^2 + DC^2$; but $AC^2 + CB^2 = 2\,AD^2 + 2\,DC^2$ (II. X., Ex. 2). Hence $AC^2 + CB_2 = 2\,S^2$.

(3′) Let AB be the base, X the vertical $\angle$, and D^2 the difference of the squares of the sides.

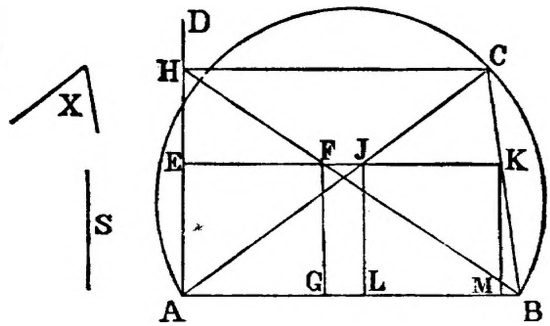


Sol.—On AB describe a segment ACB containing an $\angle$ = X. Divide AB in E, so that $AE^2 - EB^2 = D^2$ ("Sequel," Book I., Prop. IX.). Erect EC $\perp$ to AB, and join AC, BC. ACB is the triangle required.

Dem.—$AC^2 = AE^2 + EC^2$, and $BC^2 = BE^2 + EC^2$; $\therefore AC^2 - BC^2 = AE^2 - EB^2 = D^2$.

(4) Let AB be the base, X the vertical $\angle$, and S the side of the inscribed square.

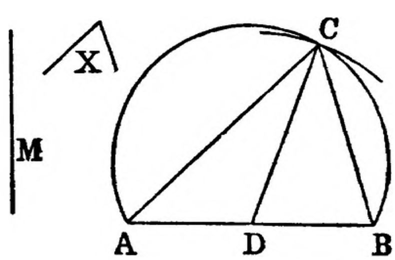
Sol.—On AB describe a segment containing an $\angle = X$. Erect



AD $\perp$ to AB. In AD take AE = S. On AE describe a square AEFG. Join BF, and produce it to meet AD in H. Through H draw HC $\parallel$ to AB, meeting the $\odot$ in C. Join AC, BC. ACB is the required triangle.

Dem.—Produce EF to meet AC, BC in J, K; and draw JL, KM $\parallel$ to AE. Now, JK = EF (I. xxxviii., Ex. 6); but EF = AE = JL; $\therefore$ JK = JL; hence the sides of JKLM are equal, and the $\angle^s$ are right (const.); $\therefore$ it is a square, and is inscribed in the $\triangle$ ABC.

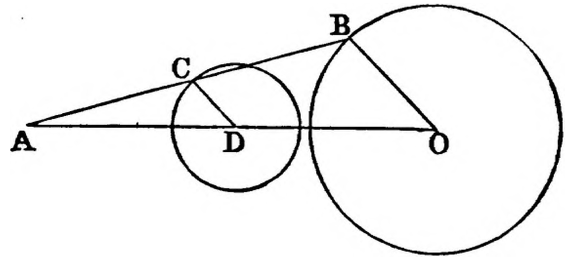
(5) Let AB be the base, M the median, and X the vertical angle.



Sol.—On AB describe a segment ACB containing an $\angle = X$; bisect AB in D. With D as centre, and a radius equal to M, describe a $\odot$, cutting ACB in C. Join AC, BC, DC. ACB is the triangle required.

Dem.—Because D is the centre of the $\odot$ cutting ACB, DC is the radius; but the radius is equal to M; $\therefore$ DC = M, and it is the median bisecting the base AB.

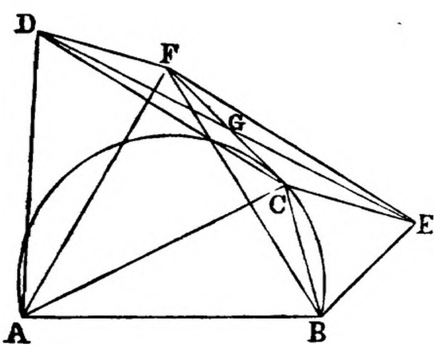
2. Let A be the fixed point, and O the centre of the given circle. Take any point B in the circumference of the ⊙. Join AB, and bisect it in C. It is required to prove that the locus of C is a circle.



Dem.—Join AO, OB, and through C draw CD ∥ to OB.

Now AO is bisected in D (I. XL., Ex. 3); but A and O are given points, ∴ the point D is given; and since CD is parallel to OB, ∴ CD = $\frac{1}{2}$ OB; but OB is a given line; ∴ CD is given, and the point D is given. Hence the locus of C is a ⊙, having D as centre and DC as radius.

3. Let AB be the base, and ACB the vertical ∠. About ACB describe a segment of a circle containing an ∠ = ACB; then the

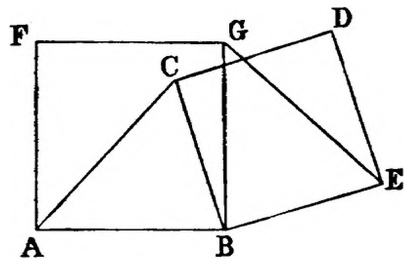


circle must pass through C. On AC, BC describe equilateral △s ADC, BEC. Join DE. It is required to find the locus of the middle point of DE.

Dem.—On AB describe an equilateral △ AFB. Join CF, DF, EF. Now the ∠ BAF = DAC, ∴ the ∠ BAC = DAF; and since DA = AC, and BA = AF, we have DA and AF equal AC and AB, and the contained ∠s are equal; hence (I. IV.) DF = CB = CE. Similarly, DC = EF; ∴ DCEF is a parallelogram; hence (I. XXXIV., Ex. 1) DE, CF bisect each other in G. Now F is a given point, and C a point on the circumference of the

circle, and FC is bisected in G; ∴ (XXXIII., Ex. 2) the locus of G is a circle.

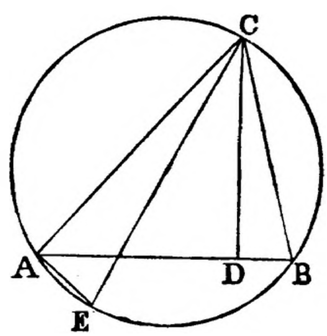
4. Let ACB be a △, whose base and vertical ∠ are given. On BC describe a square BEDC. It is required to find the locus of



E. On AB describe a square ABGF. Join EG. Now AB and BC = GB and BE, and the contained ∠s are equal; ∴ (I. IV.) the ∠ ACB = BEG; ∴ BEG is a given ∠, and the base BG is given, since it is equal to AB; ∴ (XXI., Cor. 2) the locus of E is a circle.

PROPOSITION XXXV.

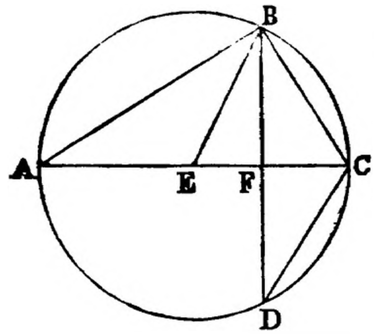
1. Let ACB be the triangle. About ACB describe a circle. Draw the diameter CE, and from C let fall a ⊥ CD on AB. It is required to prove that AC . CB = CD . CE.



Dem.—Join AE. Now the ∠ CAE is right (XXXI.), and is equal to CDB, and the ∠ AEC = ABC (XXI.); ∴ (I. XXXII., Cor. 2) the ∠ ACE = BCD, and hence (XXXV., Cor. 3) AC . CB = CD . CE.

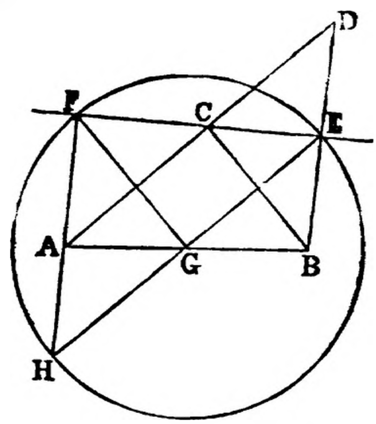
2. Let ABD be a circle, of which AC is the diameter; let AB be the chord of an arc, then BC is the chord of its supplement. Join B to the centre E. Let fall a ⊥ BF on AC, and produce it

to meet the circumference in D. It is required to prove that AB . BC = BE . BD.



Dem.—Join CD. Now the ∠ BDC = BAC (xxi.); but BAC = ABE, and BDC = DBC; ∴ ABE = DBC; hence the △s ABE, DBC are equiangular; and ∴ (xxxv., Cor. 3) AB . BC = BE . BD.

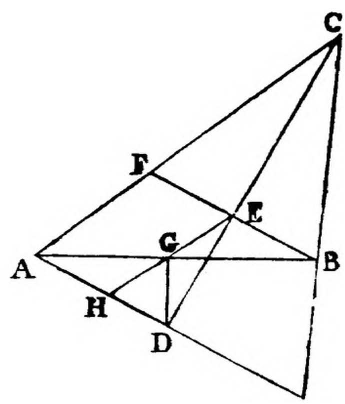
3. Let ABC be a triangle whose base and the sum of whose sides are given. Produce AC to D, and bisect the ∠ BCD by EF. From A, B let fall ⊥s AF, BE on EF. It is required to prove that AF . BE is given.



Dem.—Produce BE to meet AD. Bisect AB in G. Join EG, FG. Now because the ∠ BCE = DCE, and CEB = CED, each being right, and CE common, ∴ (I xxvi.) BE = DE, and BC = DC. Now, since BC = DC, ∴ AD = AC + CB; hence AD is given; and because AB, DB are bisected in G, E, ∴ GE is ∥ to AD, and equal to half AD (I. xl., Exs. 2 and 5); that is, $= \frac{1}{2}$ (AC + CB). Similarly, GF $= \frac{1}{2}$ (AC + CB); ∴ the ⊙, with G as centre, and GE as radius, will pass through F, and will be a given ⊙. Produce EG to meet the circumference in H, and join AH. Now because AG = GB, and GH = GE, and the

$\angle$ AGH = BGE, $\therefore$ (I. IV.) AH = BE, and the $\angle$ GAH = GBE. To each add the $\angle$ GAF, and we have the $\angle^s$ GAH, GAF = GBE, GAF; but GBE, GAF are equal to two right $\angle^s$, since BE and AF are parallel, $\therefore$ GAH and GAF are equal to two right $\angle^s$; hence AH, AF are in the same straight line. Now FGH is an isosceles $\triangle$, $\therefore$ (II. VI., Ex. 6) AF . AH = $FG^2 - AG^2$; but FG is given, since it is half the sum of AC and CB; and AG is given, because it is half AB. Hence AF . AH s given; that is, AF . BE is given.

4. Let ABC be a triangle whose base AB, and the difference of whose sides AC, CB is given. Bisect the $\angle$ ACB by CD. From



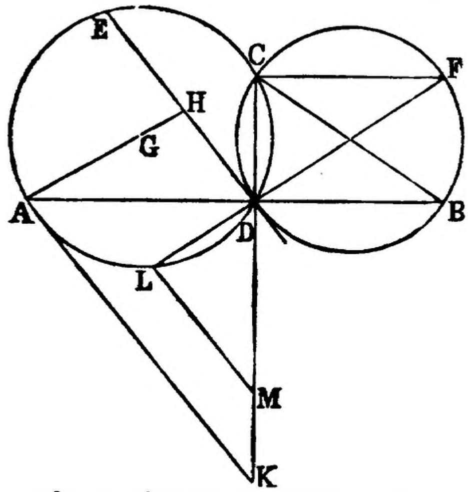
A, B let fall the $\perp^s$ AD, BE on CD. It is required to prove that AD . BE is given.

Dem.—Produce BE to meet AC in F. Bisect AB in G. Join EG, and produce it to meet AD in H. Join GD. Now because the $\angle$ BCE = FCE, and the $\angle$ BEC = FEC, and CE common; $\therefore$ (I. XXVI.) CB = CF, and EB = EF, $\therefore$ AF is the difference between AC and BC; and because EB = EF, and GB = GA, GE is $\parallel$ to AF, and equal to half AF (I. XL., Exs. 2 and 5) or half EH; $\therefore$ GE = GH; and the three lines HG, EG, DG are equal (I. XII., Ex. 2); $\therefore$ the $\triangle$ HGD is isosceles; hence (II. VI., Ex. 6) AD . AH = $AG^2 - GH^2$; but AG is given, since it is half AB; and GH is given, because it is equal EG = $\frac{1}{2}$ AF; $\therefore$ AD . AH is given; that is, AD . BE is given.

5. Let ACD, BCD be two circles intersecting in C, D. At D draw a tangent to the $\odot$ BCD, meeting ACD in E. From G, the centre of ACD, let fall a $\perp$ GH on DE, and let it meet ACD in

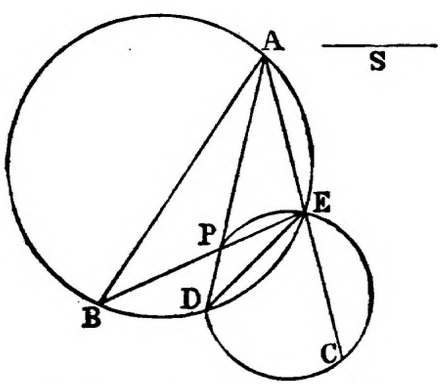
A. Join AD, and produce it to meet BCD in B. AB is the required line.

Dem.—Draw AK ∥ to DE. Join CD, and produce it to meet



AK. Take any other point L in ACD, and draw LM ∥ to DE. Join LD, and produce it to meet BCD in F. Join CF, CB. Now the ∠ EDC = CBD (xxxii.); but EDC = AKC (I. xxix.), ∴ AKC = CBD, ∴ AKBC is a cyclic quadrilateral; hence (xxxv., Cor. 3) AD . DB = CD . DK. In like manner LMFC is a cyclic quadrilateral; ∴ LD . DF = CD . DM; but CD . DK is greater than CD . DM, ∴ AD . DB is greater than LD . DF.

8. Let AB, AC be two lines given in position, and P a given point. It is required through P to draw a transversal, so that PE . PB = S^2.



Sol.—Join AP, and produce it to D, so that AP . PD = S^2. On PD describe a segment of a ⊙ PED, cutting AC in E, and containing an ∠ = BAD. Join ED, EP, and produce EP to meet AB. EPB is the required line.

Dem.—Because the $\angle$ PED = BAD, AEDF is a cyclic quadrilateral; $\therefore$ (xxxv., Cor. 3) EP.PB = AP.PD; but AP.PD is equal to S^2. Hence EP.PB is equal to S^2.

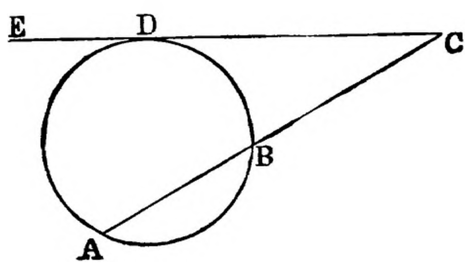
PROPOSITION XXXVI.

1. **Dem.**—Describe a circle about the triangle ACB; then because the $\angle$ BAD = ACB, AD is a tangent (xxxii.). Hence (xxxvi.) DB.DC = DA^2.

PROPOSITION XXXVII.

1. (1) Let A, B be the given points, and EC the given line. It is required to describe a $\odot$ passing through A, B, and touching the line EC.

Sol.—Join AB, and produce it to meet EC. Find a point D in EC, so that CD^2 = AC . CB; and through the points A, B, D



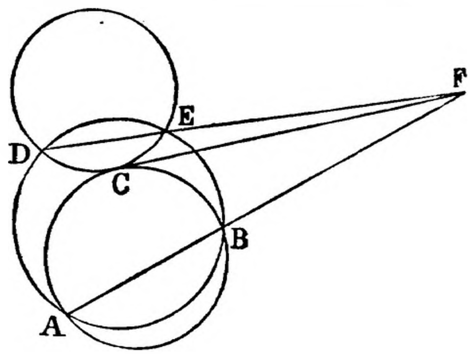
describe a circle. Then because CD^2 = AC . CB, the line CE touches the circle.

(2) Let A, B be the given points, and CDE the given circle. It is required to describe a $\odot$ passing through A, B, and touching CDE.

Sol.—Describe a $\odot$ passing through A, B, and cutting CDE in D, E. Join AB, DE, and produce them to meet in F. From F draw a tangent FC to CDE, and through the points A, B, C describe a circle. ABC is the required circle.

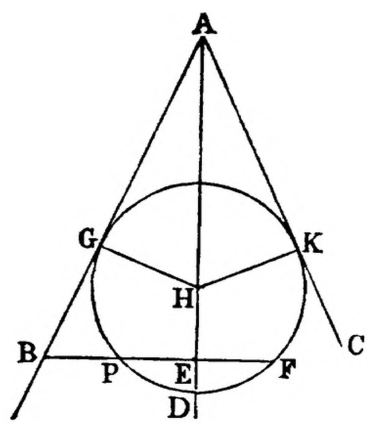
Dem.—DF . FE = AF . FB; but DF . FE = FC^2; $\therefore$ AF . FB

$= FC^2$; ∴ CF touches ABC, and it touches CDE. Hence ABC touches CDE, and it passes through A and B.



2. (1) Let AB, AC be the given lines, and P the point. It is required to describe a ⊙, touching AB, AC, and passing through P.

Sol.—Bisect the ∠ BAC by AD. From P let fall a ⊥ PE on AD, and produce it until EF = EP; let it meet AB in B. In AB



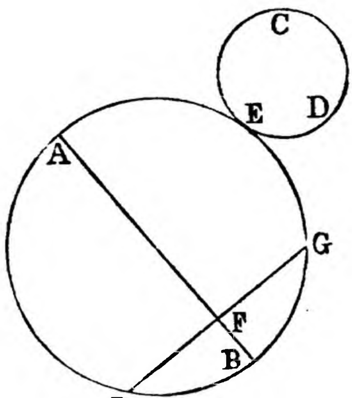
take a point G, so that $BG^2 = FB \, . \, BP$. Erect GH ⊥ to AB; and from H, where GH meets AD, let fall a ⊥ on AC. The ⊙ through the points F, P, G will be the required circle.

Dem.—Because $BG^2 = FB \, . \, BP$, the ⊙ passing through F, P, G must touch AB; and since AB touches the ⊙, and GH is ⊥ to AB, GH passes through the centre (XIX.), and AD passes through the centre (III.); ∴ H is the centre, and (I. XXVI.) the △s AGH, AKH are equal in every respect; ∴ HG = HK; but HG is the radius; ∴ HK is the radius. Hence the circle must touch AC in K.

3. Let AB be the line, CDE the ⊙, and P the point.

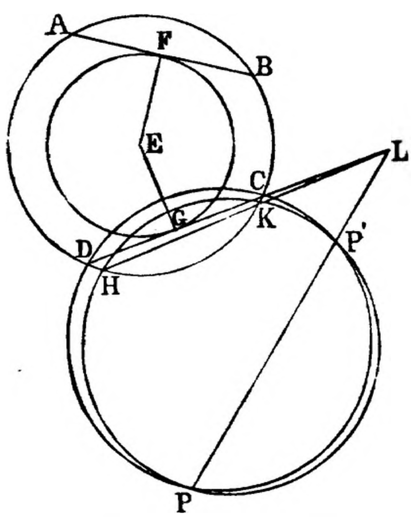
Sol.—From P let fall a perpendicular PF on AB, and produce it until FG = PF; and through P and G describe a circle PEG, touching CDE (Ex. 1). PEG is the required circle.

Dem.—Because PG is bisected at right $\angle^s$ by AB, the centre of PEG is in AB (III.)



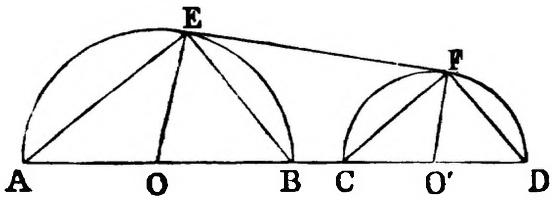
4. Let ABCD be the given $\odot$, P, P′ the points.

Sol.—Draw a line AB, cutting off an arc AB in ABCD equal to the given arc. Let E be the centre. From E draw EF $\perp$ to AB. With E as centre, and EF as radius, describe a $\odot$ FG. Through P, P′ describe a $\odot$ PP′HK, cutting ABCD in H, K. Join KH, PP′, and produce them to meet in L. Through L draw LCGD, a tangent to FG, and cutting ABCD in C, D. The $\odot$ through P, P′, C will be the required one.



Dem.—Join EG. Now because PP′KH and DCKH are cyclic quadrilaterals, $PL.LP' = HL.LK = DL.LC$; hence PP′CD is a cyclic quadrilateral; $\therefore$ the $\odot$ through P, P′, C must pass through D; and since E is the centre of FG, $EF = EG$; $\therefore AB = CD$ (XIV.), and therefore the arc $AB = CD$. Hence through P, P′ we have described a $\odot$ PP′CD, intercepting an arc $CD = AB$, on a given $\odot$ ABCD.

5. **Dem.**—Let O, O′ be the centres. Join OE, O′F. Now since OE, O′F are each $\perp$ to EF, they are $\parallel$ to each other; hence the $\angle$ DOE = DO′F; but the $\angle$ BOE is (III. xx.) double of the



$\angle$ BAE, and DO′F is double of DCF; hence the $\angle$ BAE = DCF. In like manner, the $\angle$ ABE = CDF. Hence the Δ^s ABE, CDF are equiangular.

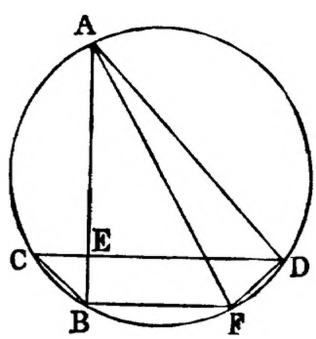
6. If r be the radius of the inscribed circle of a right-angled triangle, by making the construction, we see at once that $2r$ is equal to the excess of the sum of the legs above the hypotenuse.

Again, if ρ, ρ' be the radii of circles touching the hypotenuse, the $\perp$ from the right angle on the hypotenuse, and the $\odot$ described about the right-angled Δ, it follows at once from the Demonstration, Book VI., Ex. 59, that $\rho + \rho'$ is equal to the same excess. Hence $2r = \rho + \rho'$.

Miscellaneous Exercises on Book III.

1. Let AB, CD be two chords of a circle intersecting at right $\angle^s$. It is required to prove that the sum of the squares of the four segments is equal to the square of the diameter.

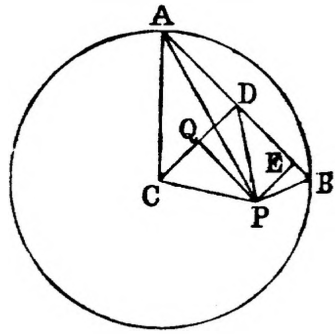
Dem.—Draw BF $\parallel$ to CD. Join CB, FD, AF, AD. Now $CB^2 = CE^2 + EB^2$; but CB = FD (xxvi., Cor. 2); $\therefore$ $FD^2 = CE^2 + EB^2$, and $AD^2 = AE^2 + ED^2$; $\therefore$ $AD^2 + FD^2 = AE^2 + EB^2$



$+ CE^2 + ED^2$; but since the $\angle$ ABF = AED (I. xxix.), $\therefore$ ABF is a right $\angle$; hence AF is the diameter; $\therefore$ the $\angle$ ADF is right; $\therefore$ $AF^2 = AD^2 + DF^2 = AE^2 + EB^2 + CE^2 + ED^2$.

2. (1) Let AB, a chord of a given $\odot$, subtend a right $\angle$ at a

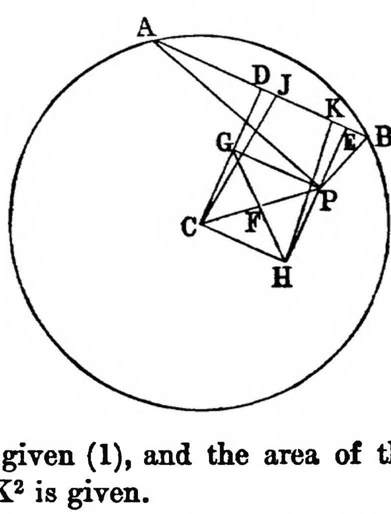
fixed point P. From P, and C, the centre of the ⊙, let fall ⊥s PE, CD on AB. It is required to prove that CD . PE is constant.



Dem.—Join CP, CA, PD, and let fall a ⊥ PQ on CD. Now AB is bisected in D (III.); ∴ the lines AD, DP, DB are equal (I. XII., Ex. 2), and $AC^2 = AD^2 + DC^2 = DC^2 + DP^2$; but $DC^2 + DP^2$ is greater than CP^2 by 2 CD . DQ (II. XIII.); that is, by 2 CD . PE; ∴ AC^2 is greater than CP^2 by 2 CD . PE; but AC^2 and CP^2 are given; ∴ CD . PE is given.

(2) Join CP, and bisect it in F. Erect FG ⊥ to CP, and equal to CF or PF. Produce GF to H, so that FH = FG, and join CG, PG, CH, PH. From C, G, H, P let fall ⊥s CJ, GD, HK, PE on AB. It is required to prove that $GD^2 + HK^2$ is constant.

Dem.—Because CGPH is a square, $GD^2 + HK^2$ is greater than 2 CJ . PE, by the area of CGPH ("Sequel," Book II., Prop. VIII.);



but 2 CJ . PE is given (1), and the area of the square is given. Hence $GD^2 + HK^2$ is given.

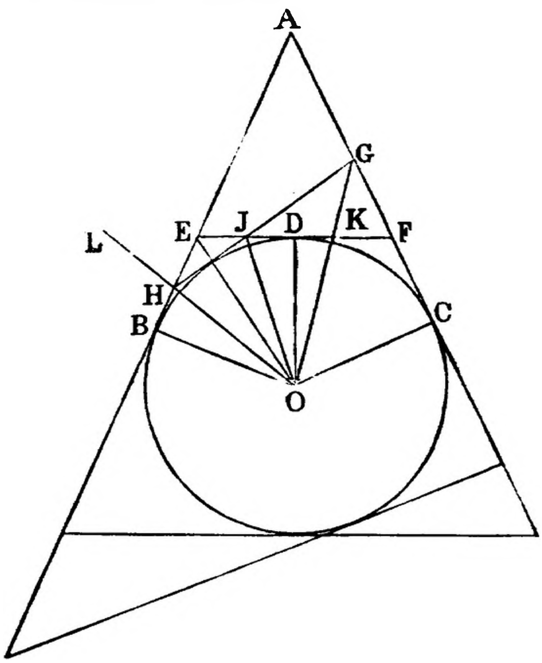
3. Let the ⊙s intersect in A, B. Through B draw a line BCD, meeting the ⊙s in C, D. Join AC, AD. It is required to prove that AC = AD.

Dem.—Because the ⊙s are equal, the arcs AB are equal;

∴ the ∠^{s} ACB, ADB are supplemental. Hence the ∠^{s} ACD, ADC are equal. And hence AC = AD.

4. (1) Let AB, AC be two fixed tangents, and EF a tangent cutting off with AB, AC an isosceles △ AEF. AEF is greater than any other △ AHG, made by a tangent HG, which does not cut off an isosceles △ with AB, AC.

Dem.—Let EF, HG intersect in J. Join OJ, OB, OC, OD, OG, OH, and produce OH to L. Now, because AB = AC, and AE = AF; ∴ BE = CF; but BE = DE, and CF = DF; ∴ DE = DF; ∴ JF is greater than JE.



Again, the ∠ HOG = BOD, because each = ½ BOC (XVII.), Ex. 9), and HOJ = ½ BOD; ∴ HOJ = JOG, and the ∠ HJO = KJO, and JO common; ∴ (I. XXVI.) JH = JK. Now the ∠ LHG is greater than HGO; but LHG = GKJ, because they are the supplements of the equal ∠^{s} OHG, OKG; ∴ GKJ is greater than JGK; ∴ JG is greater than JK; ∴ JG is greater than JH, and JF is greater than JE; ∴ the △ FJG is greater than EJH. To each add the figure AGJE, and we have the △ AEF greater than AHG.

(2) Let the tangent be drawn below the ⊙, making an isosceles △ with the fixed tangents; then it can be shown, as in (1), that the isosceles △ is less than the △ formed by any other tangent which does not cut off an isosceles △ with the fixed tangents.

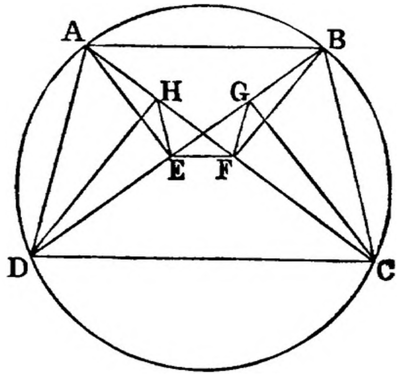
5. **Dem.**—Join CF, DE, AB. Now the ∠^{s} ADE and ABE

are equal (xxi.), and ACF, ABF equal; $\therefore$ ADE, ACF are equal; $\therefore$ CE is $\parallel$ to DF; $\therefore$ CDEF is a parallelogram, and $\therefore$ (I. xxxiv.) CD = EF.

6. See Book I., Miscellaneous Ex. 45.

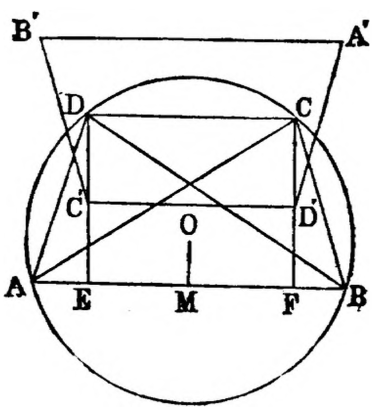
7. Let the sides of the cyclic quadrilateral ABCD be the diameters of four circles. It is required to prove that those circles intersect in four concyclic points E, F, G, H.

Dem.—Draw the diagonals AC, BD, and let fall $\perp^s$ AE, BF, CG, DH on AC, BD. Join EF, EH, GF. Now, because the $\angle^s$ AHD, CHD are right, the $\odot^s$ on AD, CD, as diameters, will pass through H. In like manner the $\odot^s$ on the other sides will pass through E, F, G. And since the $\angle^s$ AHD, AED are right,



AHED is a cyclic quadrilateral; $\therefore$ the $\angle^s$ AHE, ADE are together equal to two right $\angle^s$ (xxii.), and the $\angle^s$ AHE, FHE are equal to two right $\angle^s$; $\therefore$ the $\angle$ ADE = FHE. Similarly, BCF = EGF; but ADE = BCF (xxi.); $\therefore$ FHE = EGF. And hence (xxi., Cor. 1) the points E, F, G, H are concyclic.

8. Let ABCD be a cyclic quadrilateral. Draw the diagonals



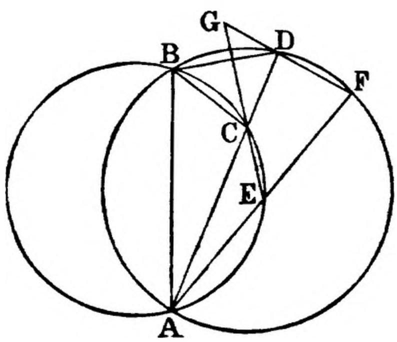
AC, BD. It is required to prove that the orthocentres of the △ˢ ADB, ACB, CAD, CBD are the angular points of a quadrilateral which is equal to ABCD.

Dem.—From D and C let fall ⊥ˢ DE, CF on AB. Let C′, D′ be the orthocentres of the △ˢ ADB, ACB; and let A′, B′ be the orthocentres of the △ˢ BCD, ADC. Join C′D′, D′A′, A′B′, B′C′; and from O, the centre, let fall a ⊥ OM on AB.

Now OM = $\frac{1}{2}$ CD′ ("Sequel," Book I., Prop. XII., Cor. 3). Similarly OM = $\frac{1}{2}$ C′D; ∴ CD′ = C′D, and they are parallel; hence DCD′C′ is a parallelogram, ∴ DC = D′C′. In a similar manner it can be shown that the other sides of A′B′C′D′ are respectively equal and parallel to the remaining sides of ABCD. Hence A′B′C′D′ = ABCD.

9. Let the circles intersect in A, B. Through A draw ACD, AEF, cutting the ⊙ˢ in C, E; D, F. Join EC, FD, and produce them to meet in G. It is required to prove that EGF is a given angle.

Dem.—Join AB, BC, BD. Now the ∠ˢ BAE, BCE are equal



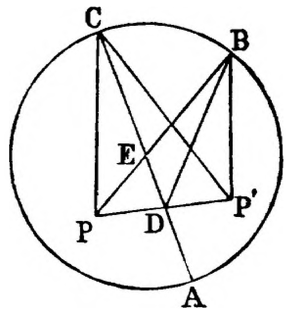
to two right ∠ˢ (XXII.), and BCE, BCG are equal to two right ∠ˢ (I. XIII.); ∴ BAE = BCG. Similarly BAE = BDG, ∴ BCG = BDG, and hence (XXI., Cor. 1) the points B, C, D, G are concyclic; ∴ the ∠ CBD = CGD. Again, the ∠ˢ ACB, ADB are given, since they are in given segments, and the ∠ CBD is equal to ACB − CBD; ∴ CBD is a given ∠; that is, CGD is a given angle.

10. See "Sequel to Euclid," Book III., Prop. x.

11. Let P, P′ be the points in the circle.

Sol.—Join PP′. Bisect it in D. Join D to the centre E, and produce it to meet the circumference in C. C is the point required.

Take any other point B in the circumference. Join BP, BP′, CP, CP′, BD. Now because E is the centre, DC is greater than DB, ∴ $2\,DC^2$ is greater than $2\,DB^2$. To each add $2\,DP^2$, and we have $2\,DC^2 + 2\,DP^2$ greater than $2\,DB^2 + 2\,DP^2$; but $CP^2 + CP'^2 = 2\,DC^2 + 2\,DP^2$ (II. x., Ex. 2), and $BP^2 + BP'^2 = 2\,DB^2 + 2\,DP^2$; ∴ $CP^2 + CP'^2$ is greater than $BP^2 + BP'^2$. Hence $CP^2 + CP'^2$ is a maximum. In like manner it can be shown, if we produce CD to A, that $AP^2 + AP'^2$ is a minimum.



12. Let ABCD (see fig., Ex. 7) be the quadrilateral. Draw AC one of the diagonals; and from B, D let fall ⊥ˢ BF, DH on AC. It is evident, from the proof of Ex. 7, that BF and DH are the common chords of the ⊙ˢ on CD, AD, and on AB, CB as diameters, and that they are parallel.

13. See "Sequel to Euclid," Book III., Prop. xi.

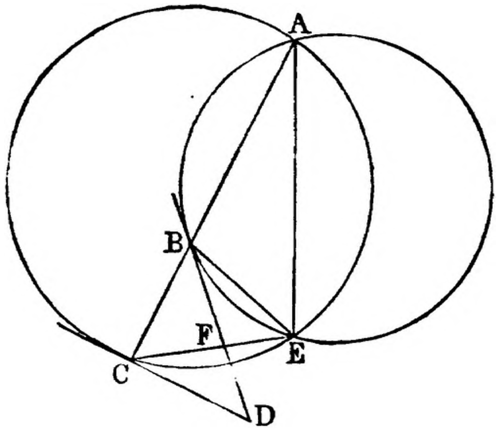
14. Let ACB be the △, and CD the internal bisector of the vertical ∠. It is required to prove that $AC \cdot CB = CD^2 + AD \cdot DB$.

Dem.—Describe a ⊙ about ACB. Produce CD to meet the circumference in E, and join BE. Now the ∠ ACE = BCE, and CAD = CEB (xxi.); ∴ (I. xxxii., Cor. 2) the △ˢ ACD, BCE are equiangular; hence (xxxv. Cor. 3) $AC \cdot CB = EC \cdot CD$; but $EC \cdot CD = ED \cdot DC + CD^2$ (II. iii.), and $ED \cdot DC = AD \cdot DB$ (xxxv.), ∴ $AC \cdot CB = CD^2 + AD \cdot DB$.

15. Draw BD, CD tangents to the circles. It is required to prove that BDC is a given angle.

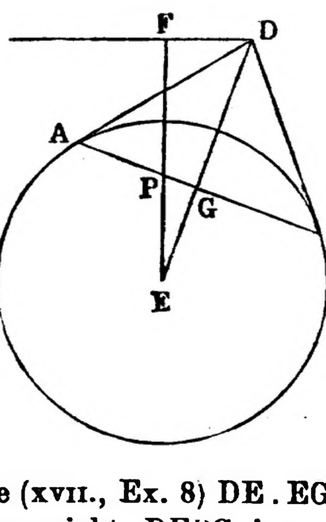
Dem.—Join AE, BE, CE. Now the ∠ DCE = CAE (xxxii.), and DBE = CAE; ∴ DCE = DBE, and CFD = BFE (I. xv.), ∴ CDF = BEF; but BEF = ABE − ACE (I. xxxii.), and

ABE and ACE are given ∠s; ∴ the ∠ BEF, that is, CDF, is given.



16. Let AB, a chord of a given circle, pass through a given point P; at A, B tangents AD, BD are drawn. It is required to prove that the locus of D is a right line.

Dem.—Let E be the centre. Join ED, EP. Produce EP, and from D draw DF ⊥ to it. Now, denoting the

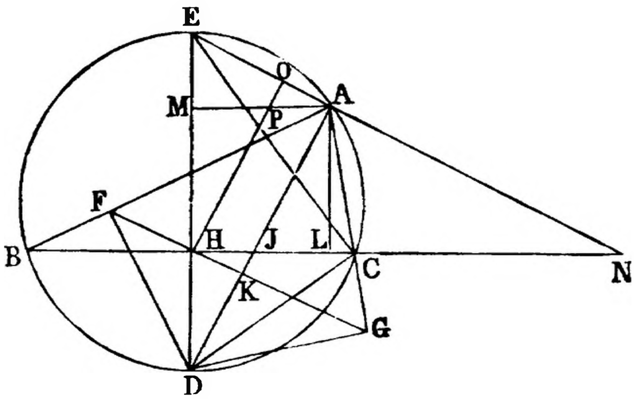


radius by R, we have (xvii., Ex. 8) DE . EG = R^2; but because the ∠s DGP, DFP are right, DFPG is a cyclic quadrilateral, and ∴ DE . EG = FE . EP; ∴ FE . EP = R^2; ∴ FE . EP is given, and EP is given; ∴ EF is given; hence F is a given point, and FD is ⊥ to EF; ∴ FD is a line given in position. Hence the locus of D is a right line.

17. Let ABC be the triangle. Describe a ⊙ about ABC. Bisect the ∠ BAC by AJ, and produce it to meet the circum-

ference in D. Through D draw the diameter DE. From A let fall a $\perp$ AL on BC. Produce AC to G; and let fall $\perp^s$ DF, DG on AB, AG; then CG = $\frac{1}{2}$(AB − AC) (Dem. of xxx., Ex. 4). It is required to prove that HJ . HL = CG².

Dem.—Join FH, GH, DC, CE, EA, and from A let fall a



$\perp$ AM on DE. Now the $\angle$ EAD is right (xxxi.), and EHJ is right, ∴ EAJH is a cyclic quadrilateral, ∴ ED . DH = AD . DJ; but because the $\angle$ ECD is right, and CH $\perp$ to ED, ED . DH = DC² (I. xlvii., Ex. 1); ∴ AD . DJ = DC², and AD . DK = DG²; hence, by subtraction, AD . JK = CG²; and since the Δ^s ADM, HJK are equiangular, we have (xxxv., Cor. 3) AD . JK = HJ . AM = HJ . HL. Hence HJ . HL = CG².

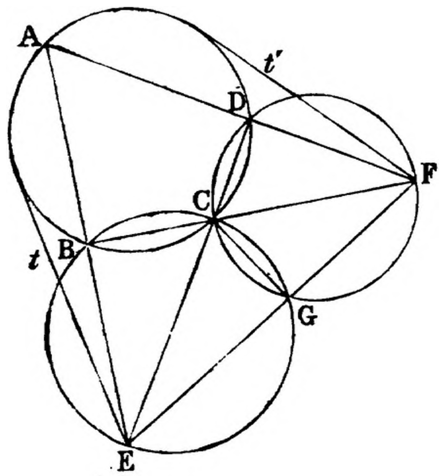
18. The rectangle contained by the distances of the point where the external bisector of the vertical $\angle$ meets the base, and the point where the $\perp$ from the vertex meets it from the middle point of the base, is equal to the square of half the sum of the sides.

Let the same construction be made as in Ex. 17. Join EA, and produce it to meet BC produced in N; then EA is the external bisector of the vertical $\angle$ (xxx., Ex. 2). It is required to prove HN . HL = AG².

Dem.—Through H draw HO ∥ to AD, meeting EN in O, and AM in P. Now the $\angle^s$ NOH, AMD are equal, each being right, and the $\angle$ PAJ = PHJ (I. xxxiv.); ∴ the $\angle$ MDA = ANH, ∴ the Δ^s HNO, AMD are equiangular, ∴ (xxxv., Cor. 3) HN . AM = DA . OH; but AM = HL, and OH = AK, ∴ HN . HL = DA . AK; but (I. xlvii., Ex. 1) DA . AK = AG². Hence HN . HL = AG².

19. Let ABCD be a cyclic quadrilateral. Produce AB, DC to meet in E, and AD, BC to meet in F. Join EF; and from E, F draw tangents t, t' to the ⊙ described about ABCD. It is required to prove that $EF^2 = t^2 + t'^2$.

Dem.—About the △ CDF describe a ⊙ CDFG, cutting EF

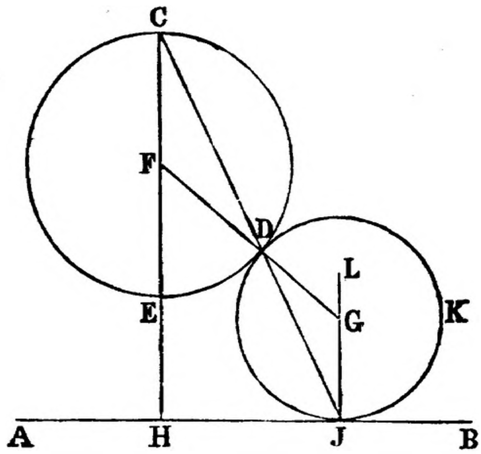


in G. Join CG. Now (XXII.) the ∠s BAD, BCD are together equal to two right ∠s, and the ∠s DFG, DCG are equal to two right ∠s; ∴ the ∠s BAD, BCD, DFG, DCG are equal to four right ∠s; and the ∠s BCD, BCG, DCG are equal to four right ∠s. Reject BCD, DCG, and we have the ∠ BCG = BAD + DFG. To each add the ∠ BEG, and we get BCG + BEG = EAF + AFE + AEF; hence the ∠s ECG, BEG are equal to two right ∠s; ∴ BCGE is a cyclic quadrilateral; ∴ FE . EG = DE . EC $= t^2$ (XXXVI.), and EF . FG = BF . FC $= t'^2$; but EF^2 = FE . EG + EF . FG; ∴ $EF^2 = t^2 + t'^2$.

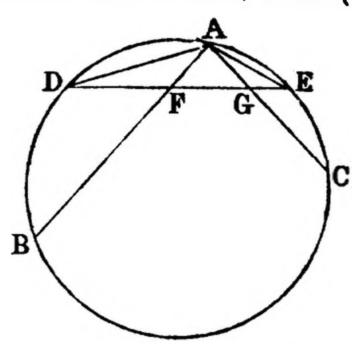
20. Let AB be a given line, CDE a given ⊙, and DKJ a variable ⊙, touching CDE in D, and AB in J. It is required to prove that JD produced passes through a given point.

Dem.—From the centre F let fall a ⊥ FH on AB, and produce it to meet the ⊙ in C. Let G be the centre of DKJ. Join FG, GJ, CD, and produce JG to L. Now (XX.) the ∠ LGD = 2 GJD = 2 GDJ, and the ∠ EFD = 2 FDC; but LGD = EFD (I. XXIX.), ∴ GDJ = FDC, ∴ JD, and DC are in one straight line; that is, the chord of contact JD produced passes through the point C

where the ⊥ from the centre of the given ⊙ on the given line meets the circumference.

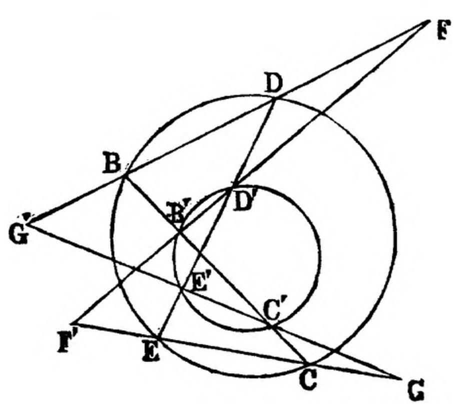


21. **Dem.**—Join DA, AE. Now the ∠ DEA = DAB (xxvii.), and EAC = ADE; but AFG = FDA + FAD (I. xxxii.) and AGF,



= GAE + GEA, ∴ AFG = AGF, and hence (I vi.) the lines AF and AG are equal.

22. **Dem.**—Join BD, B′D′, and produce them to meet in F.

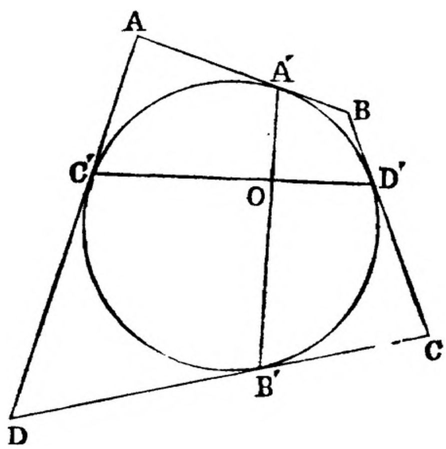


Join EC, E'C', and produce them to meet in G. Produce FB', GE to meet in F', and FB, GE' to meet in G'.

Now the ∠ BDE = BCE (xxi.), and B'D'E' = B'C'E'; but B'D'E' = DD'F (I. xv.), and B'C'E' = CC'G; hence the ∠ DFD' = CGC', and ∴ (xxi. Cor. 1) the four points F, G, F', G' are concyclic.

23. Let ABCD be a cyclic quadrilateral, such that a circle can be inscribed in it. It is required to prove that the lines A'B', C'D', joining the points of contact, are perpendicular to each other.

Dem.—Because AC' and BD' are tangents, if we produce

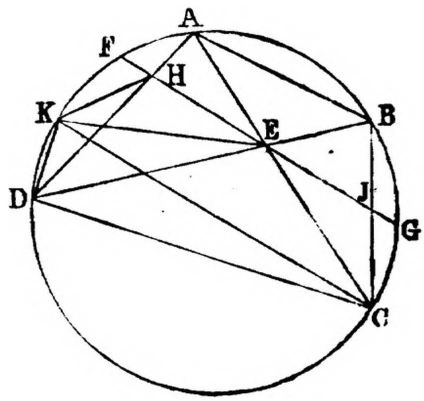


them until they meet, they will be equal, ∴ the ∠ AC'D' = BD'C'. To each add the ∠ CD'C', and we have AC'D' + CD'C' = BD'C' + CD'C'; but BD'C' + CD'C' equal two right ∠s; ∴ AC'D' + CD'C' equal two right ∠s. Similarly, AA'B' + CB'A' equal two right ∠s, and (xxii.) DAB + DCB equal two right ∠s, ∴ the sum of those six ∠s is six right ∠s; and those ∠s, together with the ∠s A'OC' + B'OD' equal eight right ∠s; ∴ A'OC' + B'OD equal two right ∠s; but A'OC' = B'OD'. Hence each is right, and therefore A'B' and C'D' are perpendicular to each other.

24. Let ABCD be a cyclic quadrilateral; AC, BD its diagonals intersecting in E. Through E draw the minimum chord FG (xv., Ex. 1). It is required to prove that EH = EJ.

Dem.—Through C draw CK ∥ to FG, and join KE, KH, KD. Now, because FG is bisected in E, and CK is ∥ to FG, ∴ EC

= EK, and the ∠ JEC = HEK; but JEC = ECK; ∴ HEK = ECK; but ECK = ADK (xxi.); ∴ HEK = ADK, and

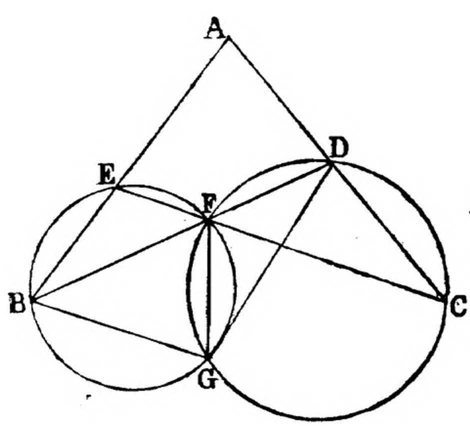


∴ HEDK is a cyclic quadrilateral; ∴ the ∠ HDE = HKE; but HDE = ACB (xxi.); ∴ HKE = ACB. And the △ˢ EHK, EJC have two ∠ˢ and a side in one equal to two ∠ˢ and a side in the other. Hence (I. xxvi.) EH = EK.

25. See "Sequel to Euclid," Book VI., Sec. i., Prop. xv. (3).

26. See "Sequel to Euclid," Book III., Prop. xx., Cor. 2.

27. Let AB, AC, BD, CE be four lines forming four △ˢ ABD, ACE, BEF, DCF. About the △ˢ BEF, DCF two ⊙ˢ are de-



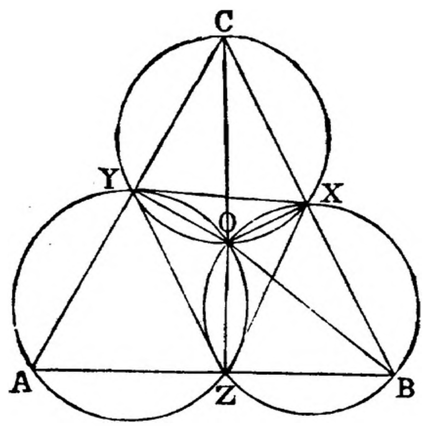
scribed intersecting in F, G. It is required to prove that the ⊙ˢ about the △ˢ ABD, ACE will pass through G.

Dem.—Join GB, GF, GD. Now the ∠ BEF = BAC + ACE; but ACE = FGD (xxi.); ∴ BEF = BAC + FGD; ∴ BEF

+ BGF = BAD + BGD; but (XXII.) BEF + BGF equal two right ∠s; ∴ BAD + BGD equal two right ∠s; hence the ⊙ about BAD will pass through G. Similarly the ⊙ about ACE will pass through G.

28. About AYZ, CXY describe ⊙s intersecting in O. It is required to prove that the ⊙ about BXZ will pass through O.

Dem.—Join OX, OY, OZ. Now the ∠s ZAY + ZOY equal two right ∠s (XXII.), and YCX + YOX equal two right ∠s; ∴ those four ∠s equal four right ∠s, and the three ∠s ZOY,



YOX, XOZ equal four right ∠s; hence the ∠ XOZ = ZAY + YCX; ∴ ZOX + ZBX = BAC + ACB + CBA, and ∴ equal two right ∠s. Hence the ⊙ about ZBX will pass through O.

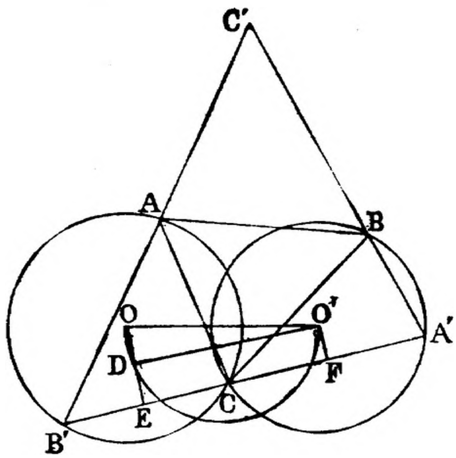
29. **Dem.**—Join OC, OB. Now, because the points O and G are given, the line OC is given in position, and YC is given in position; ∴ the ∠ YCO is given; ∴ (XXI.) the ∠ YXO is given. In like manner OXZ is given; hence the ∠ YXZ is given. Similarly, it can be shown that the ∠s XZY and XYZ are each given.

30. Let XYZ be a given △, and A, B, C three given points. It is required to place a △ equal to XYZ whose sides shall pass through A, B, C.

Sol.—Join AB, AC, BC. On BC, AC describe segments containing ∠s respectively equal to the ∠s X, Y. Join O, O′, the centres. On OO′ describe a semicircle, and in it place a chord O′D = $\frac{1}{2}$ XY. Through C draw A B ∥ to O′D. Join B A,

A′B, and produce them to meet in C′. A′B′C′ is the required triangle.

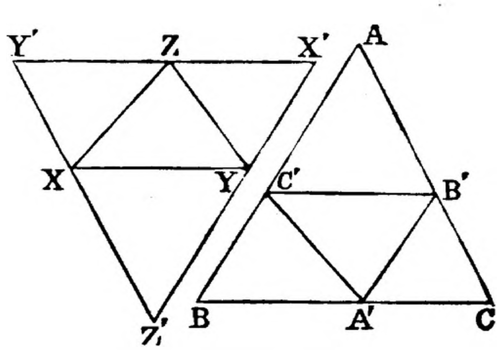
Dem.—From O′ let fall a ⊥ O′F on A′B′. Join OD, and produce it to meet A′B′ in E. Now the ∠ ODO′ is right (xxxi.);



∴ OEF is right; hence (iii.) B′C is bisected in E, and CA′ is bisected in F; ∴ B′A′ = 2 EF = 2 O′D = XY; and since the ∠s A′, B′ = X, Y respectively, the Δ A′B′C′ = XYZ.

31. Let XYZ be the given Δ, and AB, AC, BC the given lines. It is required to place a Δ equal to XYZ whose vertices shall be on AB, AC, BC.

Sol.—Through the points X, Y, Z describe a Δ X′Y′Z′ equal to ABC (Ex. 30), and in BC take BA′ = Y′Z; in BA take BC′

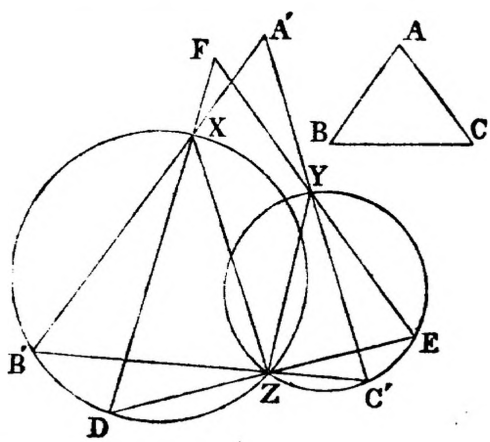


= Y′X, and in AC take B′C = X′Y. Join A′B′, B′C′, C′A′. A′B′C′ is the Δ required.

Dem.—Because A′B = Y′Z, and BC′ = XY′, and the ∠ A′B′C′ = XY′Z ; ∴ (I. iv.) A′C = XZ. Similarly A′B′ = YZ, and B′C′ = XY. Hence the △ A′B′C′ = XYZ.

32. Let ABC be the given △, and X, Y, Z the three points. It is required to construct the greatest △ equiangular to ABC, whose sides shall pass through X, Y, Z.

Sol.—Join XZ, YZ, and on them describe segments of ⊙s containing ∠s respectively equal to the ∠s B, C. Through Z draw



B′C′ ∥ to the line joining the centres. Join B′X, C′Y, and produce them to meet in A′. A′B′C′ is the △ required.

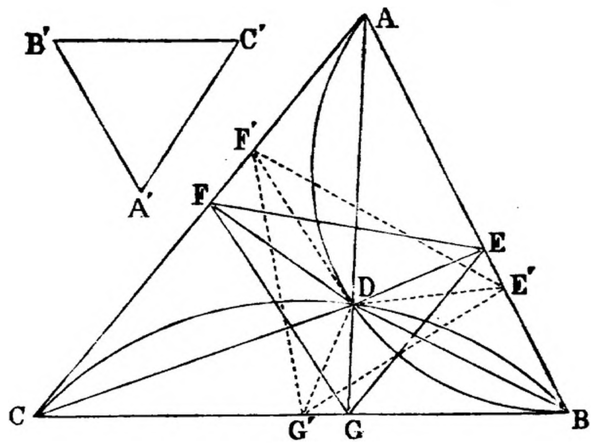
Dem.—Through Z draw any other line DE. Join DX, EY, and produce them to meet in F. Now (xxi.) the ∠ EDF = C′B′A′, and the ∠ DEF = B′C′A′, and the side B′C′ greater than DE ("Sequel," Book III., Prop. xv.). Hence the △ A′B′C′ is greater than DEF.

33. Let AB, AC, BC be the three given lines, and A′B′C′ the given △. It is required to construct the minimum △ equiangular to A′B′C, whose vertices shall be on AB, AC, BC.

Sol.—On BC describe a segment of a ⊙ containing an ∠ equal to the sum of the ∠s A, A′. On AB describe a segment containing an ∠ equal to the sum of the ∠s C, C′. From D let fall ⊥s DE, DF, DG on AB, AC, BC. Join EF, GF, EG. EFG is the required triangle.

Dem.—The ∠ CDB = A + A′ (const.); but CDB = A + DCF + DBE ; ∴ A′ = DCF + DBE. Again (const.), FCGD and EBGD

are cyclic quadrilaterals; ∴ the ∠ FCD = FGD, and DBE = DGE; hence the ∠ FGE = FCD + DBE; hence the ∠ FGE = A′. Simi-



larly GFE = B′, and GEF = C′. Therefore the △ FGE is equiangular to A′B′C′.

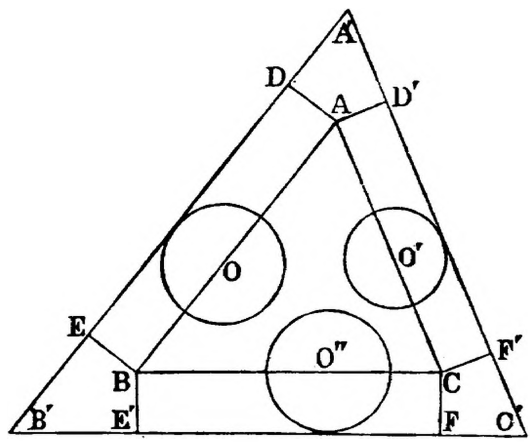
Draw any line DG′, and draw DF′, DE′, making each of the ∠s FDF′, EDE′ equal to GDG′. Join G′F′, F′E′, E′G′. Now the ∠ FDF′ = GDG′. To each add FDG′, and we have the ∠ F′DG′ = FDG. To each add the ∠ F′CG, and we get F′DG′ + F′CG′ = FDG + FCG; but FDG + FCG = two right ∠s; ∴ F′DG′ + F′CG′ = two right ∠s; hence F′CGD is a cyclic quadrilateral; ∴ the ∠ F′G′D = FCD; but FCD has been shown to be equal to FGD; ∴ F′G′D = FGD. Similarly E′G′D = EGD; ∴ F′G′E′ = FGE. In like manner G′F′E′ = GFE, and F′E′G′ = FEG. Hence the △s F′E′G′, FEG are equiangular; and since DG′ is greater than DG, and DF′ greater than DF, and the ∠ G′DF′ = GDF, the side G′F′ is greater than GF; ∴ the △ F′E′G′ is greater than FEG. Hence FEG is a minimum.

34. Let O, O′, O″ be the centres of the given ⊙s. It is required to construct the greatest △ equiangular to a given one, whose sides shall touch the three circles.

Sol.—Through the points O, O′ O″, describe the maximum △ ABC, equiangular to the given one. Draw tangents A′B′, B′C′, C′A′ respectively ∥ to AB, BC, CA. A′B′C′ is the required △.

Dem.—From A, B, C let fall ⊥s on the sides of the △ A′B′C′. Because the ∠s about B are together equal to four

right ∠s; and that the ∠s EBA, E′BC are each right, the ∠s EBE′, ABC are together equal to two right ∠s; but ABC is a given ∠; ∴ EBE′ is given, and the sides BE, BE′ are given, since they are equal to the radii of the ⊙s O, O″. Hence the figure EBE′B′ is given in magnitude. Similarly the figures ADA′D′, CFC′F′ are given in magnitude. Again, since the



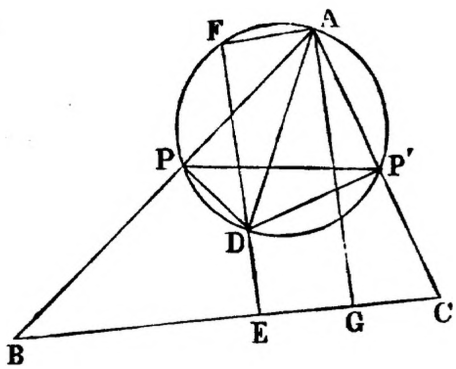
Δ ABC is a maximum, the side BC is a maximum; therefore BCFE′ must be a maximum, because it is contained by BC and BE′, which is a given line, being equal to the radius of O″. In like manner each of the figures ABDE, ACD′F′ is a maximum. Hence the whole figure A′B′C′ is a maximum.

35. Let AB, AC, two sides of a given Δ ABC, pass through two fixed points P, P′. It is required to prove that the side BC touches a fixed circle.

Dem.—Join PP′. Describe a ⊙ about the Δ APP′. Draw the diameter AD, and join DP, DP′. From D let fall a ⊥ DE on BC, and produce it to meet the ⊙ in F. Join AF, and let fall a ⊥ AG on BC.

Now since the points P, P′ are given, PP′ is a given line, and the ∠ PAP′ is given; hence (XXI., Cor. 2) the circle PAP′ is given; and because the ∠s EBP, EDP are together equal to two right ∠s, and EDP, FDP are together two right ∠s, ∴ the ∠ FDP = EBP, and is therefore a given ∠; hence the arc PF is given, and ∴ F is a given point. Again (XXXI.) the ∠ AFD is right, and FEG is right; hence AFEG is a parallelogram; ∴ EF = AG; but AG is given, since it is the ⊥ from the vertex on the base of a given Δ; ∴ EF is given, and the point F is given;

hence the locus of E is a ⊙, having F as a centre, and EF as

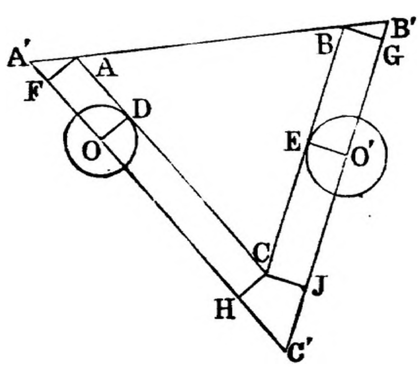


radius. Hence the base BC touches a fixed circle.

36. Let the sides CA, CB of the △ ABC touch fixed ⊙s. It is required to prove that AB touches a fixed ⊙.

Dem.—Through the centres O, O′ draw ∥s A′C′, B′C′ to AC, BC. Join O, O′ to the points of contact D, E; and through A, B, C draw ∥s AF, BG, CH, CJ to OD, O′E.

Now the ∠ BAC = B′A′C′; ∴ BA′C′ is given, and the ∠ AFA′ is right; ∴ the △ AA′F is given in species, and the side AF is given, being equal to OD; ∴ AA′, A′F are each given. Again, the ∠s ACB, HCJ are equal to two right ∠s; but ACB is given

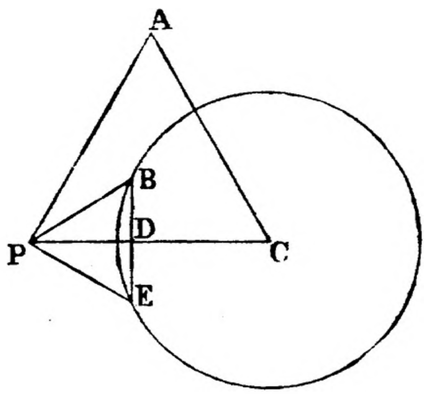


∴ HCJ is given, and the sides CH, CJ are given; ∴ HCJC′ is a given figure; ∴ C′H is given, and HF is given, being equal to AC; ∴ A′C is a given line. Similarly B′C′ is given, and A′B′ is given; ∴ the △ A′B′C′ is given. And hence (Ex. 35 A′B′ touches a fixed circle.

37. Let P be the given point, and C the centre of the circle.

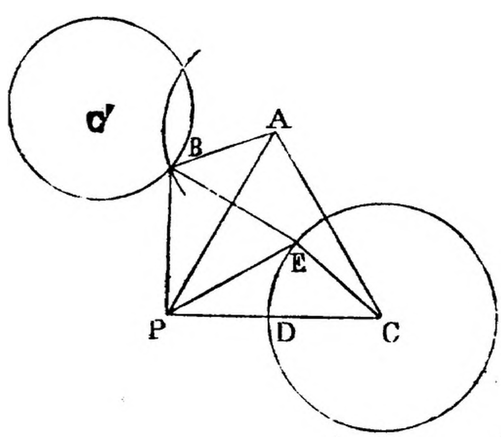
Sol.—Join PC, and on it describe an equilateral △ PAC. Draw PB, bisecting the ∠ APC. From B let fall a ⊥ BD on PC, and produce it to meet the ⊙ in E. Join EP. EPB is the required triangle.

Dem.—BD = ED (III.), and DP common, and the ∠ BDP = EDP; ∴ (I. IV.) PB = PE, and the ∠ BPD = EPD; but



BPD is ½ an ∠ of an equilateral △; ∴ EPD is ½ an ∠ of an equilateral △. Hence EPB is an ∠ of an equilateral △, and the ∠ PEB = PBE. Hence the △ EPB is equilateral.

38. Let P be the given point, and C, C′ the centres of the given ⊙s. It is required to construct an equilateral △, having its vertex at P, and the extremities of its base on the circumferences of C and C′.



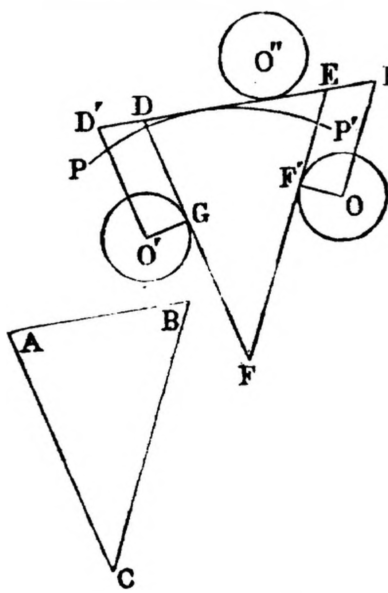
Sol.—Join PC, and on it describe an equilateral △ PAC. With A as centre, and a radius equal to CD, describe a ⊙, cut-

ting the ⊙ whose centre is C′ in B. Join AB, and at the point C in CP make the ∠ PCE = BAP (I. xxiii.). Join BE, EP, PB. BEP is the required △.

Dem.—Because AP = CP, and AB = CE, and the ∠ BAP = ECP, ∴ (I. iv.) the base BP = EP, and the ∠ BPA = CPE. To each add the ∠ APE, ∴ the angle BPE = CPA, hence BPE is an ∠ of an equilateral △. And since PB = PE, the △ PBE is equilateral.

39. Let ABC be a given △. It is required to place it so that its sides shall touch three given ⊙s O, O′, O″.

Sol.—If two sides of a △ equal to ABC touch two ⊙s O, O′, the third must touch a fixed ⊙ (Ex. 36). Let PP′ be the fixed ⊙.

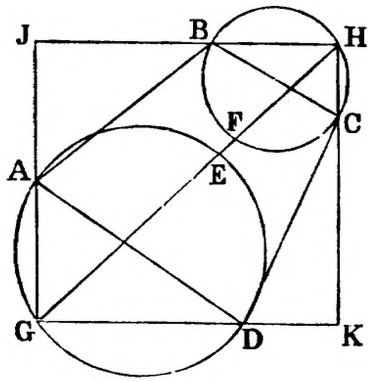


Draw DE a common tangent to O″, PP′ (xvii., Ex. 10). Through O, O′ draw OE′, O′D′, meeting DE produced, and making the ∠s OE′D′, O′D′E′ respectively equal to the ∠s CBA, CAB. At O, O′ draw OF′, OG′ at right ∠s to OE′, O′D′; and through F′, G draw EF, DF, touching the ⊙s. DEF is the △ required.

Dem.—Because each of the ∠s E′OF′, EF′O is right, E′O, EF are ∥; ∴ the ∠ DEF = DE′O, and ∴ equal CBA. Similarly, EDF is = CAB. Hence DEF is the △ required.

40. Let ABCD be a given quadrilateral. It is required to describe a square about it.

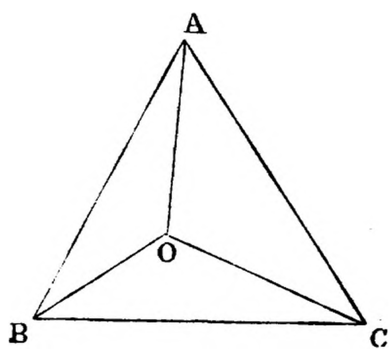
Sol.—On AD, BC, two opposite sides, as diameters, describe ⊙s AED, BFC. Bisect the semicircles AED, BFC in E, F.



Join EF, and produce to meet the ⊙s again in G, H. Join HB, GA, and produce them to meet in J. Join GD, HC, and produce them to meet in K. GJHK is the required square.

Dem.—Because the arc AE = DE, the ∠ AGE = DGE; but the ∠ AGD is right (xxxi.), ∴ AGE is half a right ∠. In like manner BHF is half a right ∠, ∴ AGE = BHF, ∴ JH = JG. Similarly, KG = KH; hence the sides are equal, and the ∠s are evidently right. Therefore GJHK is a square.

Lemma.—To find a point O in a △ ABC, such that the ∠ BOC

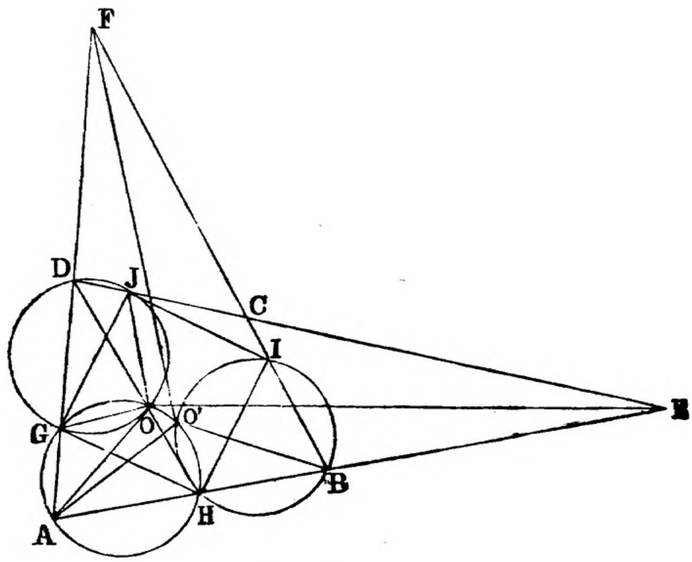


may exceed the ∠ BAC by a given ∠ X, and that the ∠ AOC may exceed the ∠ ABC by a given ∠ Y.

Sol.—On BC describe a segment of a ⊙ containing an ∠ equal to BAC + X, and on AC describe a segment containing an ∠ equal

to ABC + Y. The point O, in which these segments intersect, is evidently the required one.

41. Let ABCD be a given quadrilateral. It is required to inscribe a square in it.

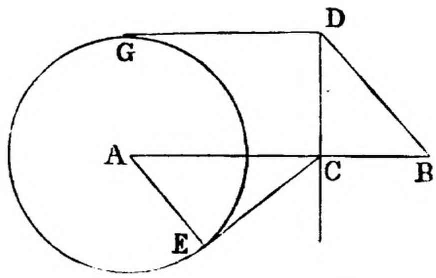


Sol.—Produce AB, DC to meet in E, and AD, BC to meet in F. In the $\triangle$ AED find a point O, such that the $\angle$ AOD is equal to AED, together with a right $\angle$, and that the $\angle$ DOE is equal to FAE, together with half a right $\angle$ (Lemma); and in AFB find a point O′, so that the $\angle$ AO′B is equal to AFB, together with a right $\angle$; and the $\angle$ AO′F = ABF, together with half a right $\angle$. Describe a $\odot$ through the points O, O′, A; cutting AF, AE in G, H. Through O, G, D describe a $\odot$, cutting DE in J; and through O′, H, B describe a $\odot$, cutting BF in I. Join GJ, JI, IH, HG. GJIH is the required square.

Dem.—Join OG, OH, OJ. Now the difference between the $\angle$s AOD and AED is equal to a right $\angle$ (const.), and AOD − AED = EAO + ODE; hence EAO and ODE are together equal to a right $\angle$; but EAO = HGO (xxi.), and ODE = OGJ, hence the $\angle$ HGJ is right. Similarly, by joining JH, it can be shown that GJH is half a right $\angle$, ∴ GH = GJ. Similarly, it can be shown that the $\angle$ GJI is right, and that GJ = JI. Hence GJIH is a square.

42. (1) *Lemma.*—To find the radical axis of a $\odot$ and a point. Let A be the centre of the $\odot$, and B the point.

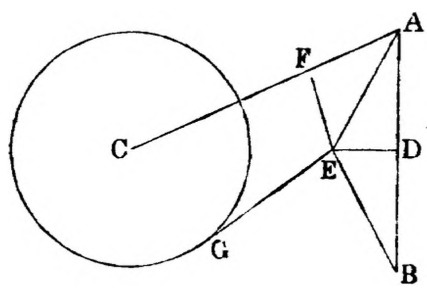
Sol.—Join AB, and divide it in C, so that $AC^2 - CB^2$ is equal



to the square of the radius AE ("Sequel," Prop. IX., p. 7). Erect CD ⊥ to AB. CD is the required radical axis.

Dem.—Draw DG a tangent from any point D. Join DB. Draw CE a tangent, and join AE. Now $AC^2 - CB^2 = AE^2$, $\therefore AC^2 - AE^2 = CB^2$; that is, $CE^2 = CB^2$, $\therefore CE^2 + CD^2 = CB^2 + CD^2$; but $CE^2 + CD^2 = GD^2$ ("Sequel," Prop. XXI., p. 42), and $CB^2 + CD^2 = DB^2$; $\therefore DG^2 = DB^2$. Hence CD is the radical axis (XVII., Ex. 6).

Sol.—Let C be the centre of the ⊙, and A, B the points. Join AB, and bisect it in D. Erect DE ⊥ to AB. Join AC,



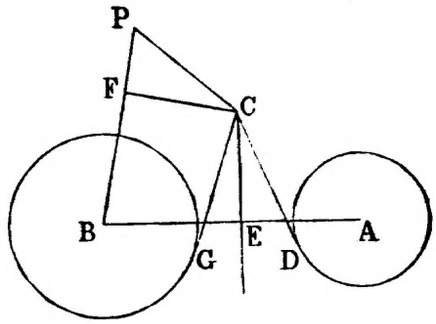
and find the radical axis FE (Lemma) of the ⊙ and the point A, and let it cut DE in E. E is the centre of the required ⊙.

Dem.—From E draw the tangent EG to the ⊙. Join EA, EB. EG = EA (Lemma), and EA = EB; ∴ EA, EB, EG are equal; and the ⊙, with E as centre and EA as radius, will pass through B, and cut the given ⊙ orthogonally in G ("Sequel," Book III., Prop. XXII.).

(2) *Lemma.*—To find the radical axis of two circles. Let A, B be the centres. Join AB, and divide in E, so that $AE^2 - EB^2$ is equal to the difference of the squares of the radii. Erect EC ⊥ to AB. From C and E draw tangents CD, EH, CG, EJ to A and B. Join AH, BJ. Now $AE^2 - EB^2 = AH^2$

$- BJ^2$, $\therefore$ $EH^2 = EJ^2$, $\therefore$ $CE^2 + EH^2 = CE^2 + EJ^2$; hence ("Sequel," Book III., Prop. XXI.) $CD^2 = CG^2$. Hence EC is the radical axis of the circles.

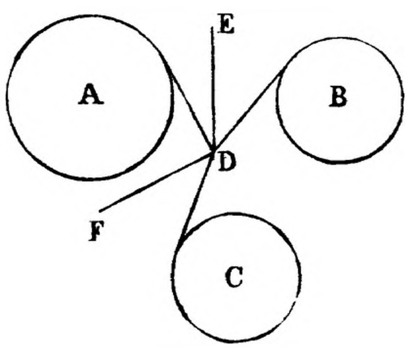
Sol.—Let A, B be the centres, and P the point. Join AB,



and find the radical axis CE (*Lemma*). Join BP, and find the radical axis CF of the point P, and the ⊙ B. From C, where CE, CF intersect, draw tangents CD, CG to A and B. Join CP. C is the centre of the required circle.

Dem.—Since CE is the radical axis of the ⊙s A, B, CG = CD (*Lemma*); and because CF is the radical axis of the ⊙ B and the point P, CG = CP; $\therefore$ CG, CD, CP are equal, and therefore the ⊙, whose centre is C, and radius CP, will cut the ⊙s A and B orthogonally ("Sequel," Book III., Prop. XXI.).

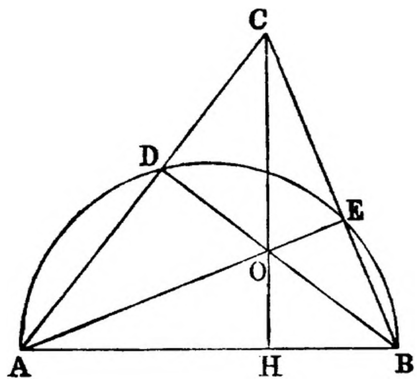
(3) Let A, B, C be the ⊙s. Find DE, the radical axis of A



and B, and DF the radical axis of A and C. From D, where DE, DF intersect, draw tangents to A, B, C. Now these tangents are equal; and the ⊙, with D as centre, and one of them as distance, will pass through the ends of the other two, and will cut the ⊙s A, B, C orthogonally ("Sequel," Book III., Prop. XXI.).

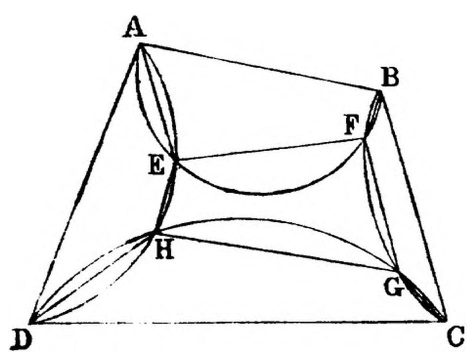
43. **Dem.**—Join BD, AE, and let them intersect in O. Join CO, and produce it to meet AB in H.

Now (xxxi.) each of the $\angle$s ADB, AEB is right, ∴ BD, AE are $\perp$s to AC, BC; hence (I., Ex. 16, Miscellaneous) CH is $\perp$ to AB. Now (xxii., Ex. 1) AHEC is a cyclic quadrilateral;



∴ (xxxvi.) BC.BE = AB.BH. And since BHDC is a cyclic quadrilateral, AC.AD = AB.AH. Adding, we get AC.AD + BC.BE = AB (AH + BH) = AB^2.

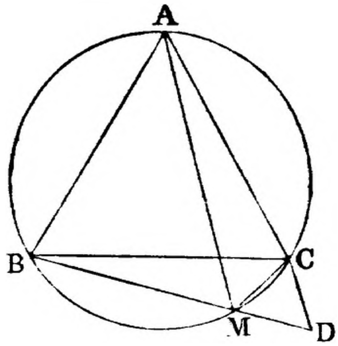
44. **Dem.**—Join AE, BF, CG, DH. Now (xxii.) the $\angle$s AEF, ABF are together equal to two right $\angle$s, and similarly the $\angle$s AEH, ADH are together equal to two right $\angle$s; hence the sum of those $\angle$s is four right $\angle$s, and the sum of the $\angle$s AEF, AEH,



FEH is four right $\angle$s; ∴ the $\angle$ FEH = ABF + ADH. In like manner the $\angle$ FGH = FBC + HDC; ∴ the $\angle$s FEH and FGH = ABC and ADC, and ∴ are equal to two right $\angle$s. Hence (xxii., Ex. 1) EFGH is a cyclic quadrilateral.

45. **Dem.**—Describe a ⊙ about ABC. Take any point M in

the circumference. Join MA, MB, MC. It is required to prove that MA = MB + MC. Produce BM to D, so that MD = MC. Join CD. Now (xxii.) the ∠s BAC and BMC are together

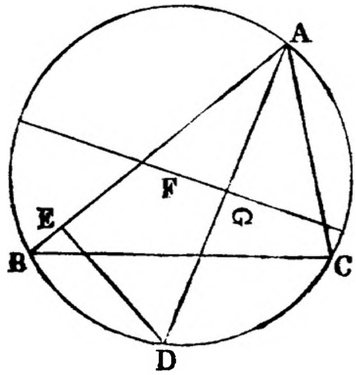


equal to two right ∠s, and BMC, DMC together equal to two right ∠s; ∴ DMC = BAC, and is an ∠ of an equilateral Δ; and because MC = MD, MCD is an equilateral Δ.

Again, because BMCA is a cyclic quadrilateral, the ∠ MBC = MAC, and ABC = AMC; but ABC = MDC, since each is an ∠ of an equilateral Δ; ∴ AMC = MDC; hence (I. xxvi.) the Δs AMC, BDC are equal; ∴ AM = BD; that is, AM = MB + MC.

46. (1) Let ABC be a Δ, the sum of whose sides AB, AC is given, and the ∠ BAC, both in magnitude and position. About the Δ ABC describe a ⊙. It is required to prove that the locus of its centre F is a right line.

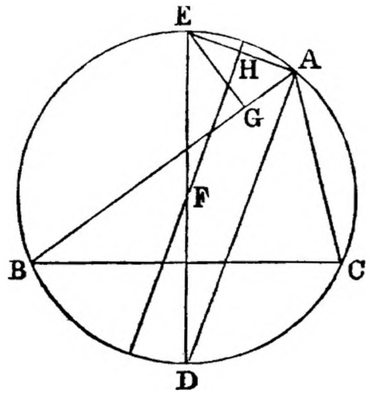
Dem.—Bisect the arc BC in D. Join AD. Let fall a ⊥ DE



on AB. From F let fall a ⊥ FG on AD. Now AE = ½ (AB

+ AC) (xxx., Ex. 4); hence AE is a given line; ∴ E is a given point. And since DE is ⊥ to AE, at a given point, DE is given in position; and because the ∠ BAD = $\frac{1}{2}$ BAC, BAD is a given ∠; ∴ AD is given in position, and DE is given in position; ∴ D is a given point, and the point A is given; hence AD is a given line, and (III.) AD is bisected in G; ∴ G is a given point, and FG is ⊥ to a line given in position; hence FG is given in position. Hence the locus of F is the line FG.

(2) Bisect the ∠ BAC by AD. Erect DE ⊥ to BC. DE is



the diameter. Join EA, and from E, F let fall ⊥s EG, FH on AB and AE.

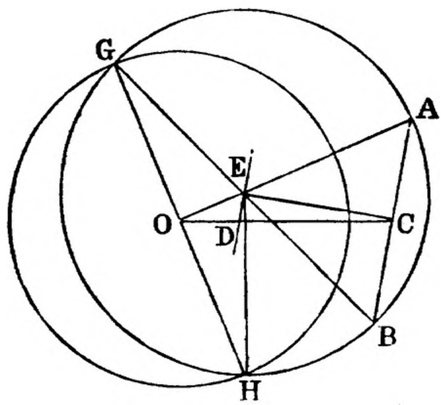
Now the line AG is given, for it is equal to $\frac{1}{2}$ (AB − AC); ∴ EG, which is ⊥ to it, is given in position, and EA is given in position, since it is ⊥ to AD; ∴ E is a given point, and EA is bisected in H (III.); ∴ FH is given in position. Hence the locus of F is the line FH.

47. (1) Let O be the centre of the given ⊙, and A, B the points. It is required to describe a ⊙ which shall pass through A, B, and bisect the circumference of the given ⊙.

Sol.—Bisect AB in C. Join CO, and divide it in D, so that $CD^2 - OD^2 = R^2 - BC^2$ (R being the radius of the given ⊙). Erect DE, CE, ⊥s to OC, AB, and join AE, BE, OE. E is the centre of the required ⊙.

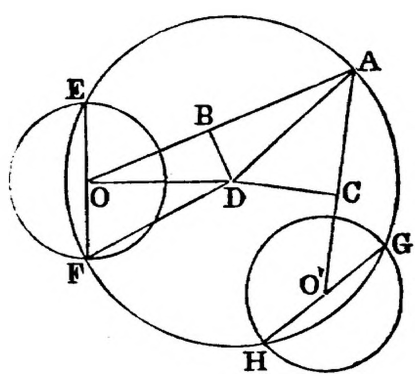
Dem.—The △s ACE, BCE are equal (I. IV.); ∴ AE = BE; hence the ⊙, with E as centre, and AE as radius, will pass through B. Let it cut the given ⊙ in G, H. Join OG, OH, EG, EH. Now $CD^2 - OD^2 = R^2 - BC^2$; ∴ $CE^2 - OE^2 = R^2$

$- BC^2$; $\therefore BC^2 + CE^2 = R^2 + OE^2$; that is, $BE^2 = R^2 + OE^2$; $\therefore GE^2 = R^2 + OE^2$; but $OG = R$; $\therefore GE^2 = OG^2 + OE^2$; hence



the $\angle$ EOG is right. Similarly EOH is a right angle; $\therefore$ OG and OH are in the same straight line; hence GH is the diameter of the given $\odot$. Hence the circumference of the given $\odot$ is bisected by the $\odot$ ABH in the points G, H.

(2) Let A be the given point, and O, O′ the centres of the given $\odot^s$. It is required to describe a $\odot$ passing through A which shall bisect the circumferences of the $\odot^s$ whose centres are O, O′.



Sol.—Join AO, and divide it in B, so that $AB^2 - BO^2 = R^2$ (R being the radius of the $\odot$ whose centre is O). Join AO′, and divide it in C, so that $AC^2 - CO'^2 = R'^2$. Erect BD, CD $\perp^s$ to AO, AO′. Join AD. With D as centre, and AD as radius,

describe a ⊙ EAG, cutting the given ⊙s in the points E, F; G, H. This is the ⊙ required.

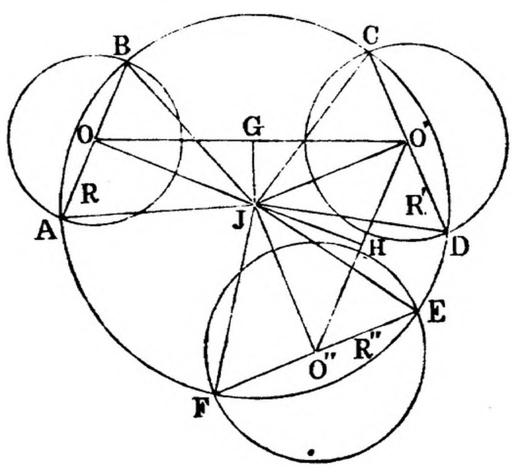
Dem.—Join OE, OF, O'G, O'H, OD, FD. Now $AB^2 - OB^2 = OF^2$ (const.), ∴ $AD^2 - OD^2 = OF^2$, ∴ $AD^2 = OD^2 + OF^2$; that is, $FD^2 = OD^2 + OF^2$, ∴ the ∠ DOF is right. Similarly, the ∠ DOE is right, ∴ OE and OF are in the same straight line. Hence EF is the diameter of one of the given ⊙s. In like manner GH is the diameter of the other given ⊙. Hence the circumferences of the given ⊙s are bisected by the ⊙ EAG.

48. Let a ⊙, whose centre is D, bisect the circumferences of two given ⊙s in the points E, F; G, H. It is required to find the locus of D. (See last diagram.)

Sol.—Join EF, GH. Now since the circumferences are bisected in E, F; G, H, the centres, must be in the lines EF, GH. Bisect these lines in O, O'. Join OO', DO, DO'. From D let fall a ⊥ DJ on OO'. DJ is the locus of D.

Dem.—Join DF, DH. Now (III.) the ∠s DOF, DO'H are right, ∴ $DF^2 = DO^2 + OF^2$, and $DH^2 = DO'^2 + O'H^2$; but $DF^2 = DH^2$, ∴ $DO^2 + OF^2 = DO'^2 + O'H^2$, ∴ $DO^2 - DO'^2 = O'H^2 - OF^2$; but $O'H^2 - OF^2$ is given, since O'H and OF are the radii of two given ⊙s, ∴ $DO^2 - DO'^2$ is given, ∴ $OJ^2 - O'J^2$ is given; ∴ J is a given point; ∴ the line DJ is given in position. Hence the locus of D is the line DJ.

49. Let O, O', O'' be the centres of the given ⊙s; and R, R',

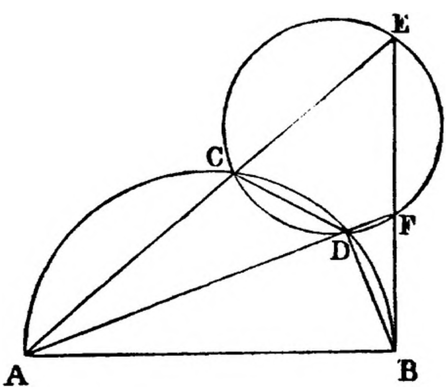


R'' their radii. It is required to describe a ⊙ which shall bisect the circumferences of the given circles.

Sol.—Join OO′, and divide it in G, so that $OG^2 - O'G^2 = R'^2 - R^2$. Join O′O″, and divide it in H, so that $O''H - O'H^2 = R'^2 - R''^2$; and at G, H erect GJ, HJ ⊥ˢ to OO′, O′O″. The point J, where these perpendiculars intersect, is the centre of the required circle.

Dem.—Join OJ, O′J, O″J. Through O, O′, O″ draw AB, CD, EF at right angles to OJ, O′J, O″J, and join JA, JB, JC, JD JE, JF. Now $OA^2 = OB^2$, $\therefore OA^2 + OJ^2 = OB^2 + OJ^2$; $\therefore AJ^2 = BJ^2$; $\therefore AJ = BJ$. In like manner $CJ = DJ$, and $EJ = FJ$. Again, $OG^2 - O'G^2 = R'^2 - R^2$; $\therefore OG^2 + R^2 = O'G^2 + R'^2$, $\therefore OG^2 + JG^2 + R^2 = O'G^2 + JG^2 + R'^2$; that is, $OJ^2 + R^2 = O'J^2 + R'^2$; $\therefore AJ^2 = DJ^2$; $\therefore AJ = DJ$. Similarly, $BJ = EJ$, and $CJ = FJ$. Hence those six lines are equal; and the ⊙, with J as centre, and AJ as radius, will pass through the points B, C, D, E, F, and will bisect the circumferences of the given circles in those points.

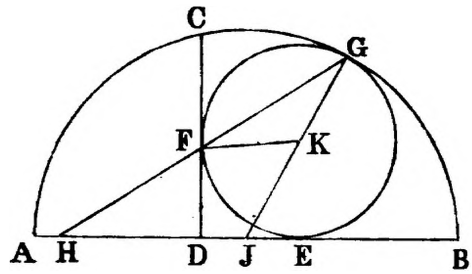
50. **Dem.**—Join BC, CD, DB. Now, since ABE is a right-angled △, and BC is ⊥ to AE, we have $AE . AC = AB^2$



(I. XLVII., Ex. 1). In like manner $AF . AD = AB^2$; therefore $AE . AC = AF . AD$. Hence the points C, E, F, D are concyclic.

51. (1) **Dem.**—Let J, K be the centres of the ⊙ˢ. Join JK, and produce it. JK produced must pass through G (XI.) Join KF. If GF does not pass through A, let it pass through H. Now (XVIII.) the ∠ CFK is right, and the ∠ CDB is right, ∴ FK and AB are parallel, ∴ the ∠ GFK = GHB; but GFK

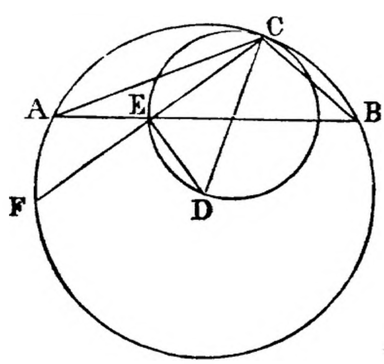
= FGK (I. v.); ∴ the ∠ JHG = JGH; hence JG = JH; but JG



= JA; ∴ JH = JA, which is absurd. Hence GF produced must pass through A.

(2) Complete the ⊙ ACD, and produce CD to meet the circumference again in M. Now (III.) DC = DM, ∴ the arc AC = AM; hence (Ex. 26) AF . AG = AC^2, and (XXXVI.) AF . AG = AE^2, ∴ $AC^2 = AE^2$; ∴ AC = AE.

52. Let ACB be an obtuse-angled triangle. It is required to draw from C a line CE, so that CE^2 = AE . EB.



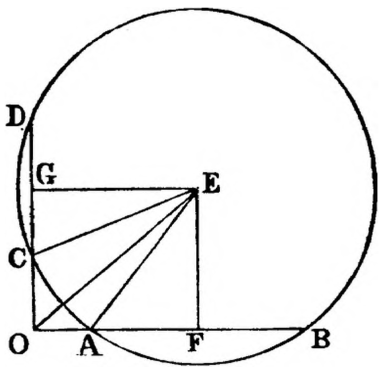
Sol.—Describe a ⊙ about ACB. Let D be its centre. Join CD. On CD as diameter describe a ⊙, cutting AB in E. Join CE. CE is the required line.

Dem.—Produce CE to meet the circumference again in F, and join DE.

Now the ∠ CED is right (XXXI.), ∴ FED is right; hence (III.) CF is bisected in E; ∴ FE . EC = EC^2; but (XXXV.) FE . EC = AE . EB, ∴ AE . EB = CE^2.

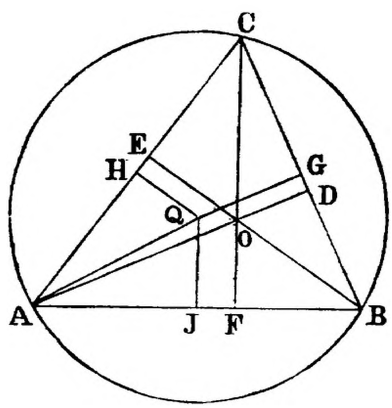
53. **Dem.**—From E let fall ⊥s EF, EG on AB, CD, and join AE, CE. Now AF = BF (III.); ∴ $AB^2 = 4\,AF^2$. Similarly, $CD^2 = 4\,CG^2$; ∴ $AB^2 + CD^2 = 4\,AF^2 + 4\,CG^2$. Again (I. XLVII.),

$OE^2 = OG^2 + EG^2 = EF^2 + EG^2$; $\therefore$ $4\,OE^2 = 4\,EF^2 + 4\,EG^2$. Adding, we get $AB^2 + CD^2 + 4\,OE^2 = 4\,AF^2 + 4\,EF^2 + 4\,CG^2$



$+ 4\,EG^2$; but $4\,AF^2 + 4\,EF^2 = 4\,AE^2 = 4\,R^2$, and $4\,CG^2 + 4\,EG^2 = 4\,CE^2 = 4\,R^2$. Hence $AB^2 + CD^2 + 4\,OE^2 = 8\,R^2$.

54. (1) Let ABC be the triangle. From A, B, C let fall $\perp^s$ AD, BE, CF on the sides, and intersecting in O. It is required to



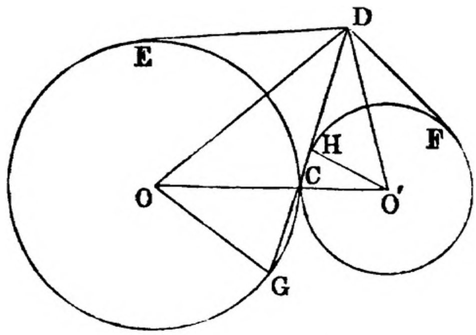
prove that $AB^2 + BC^2 + CA^2$ is equal to $2\,AO\,.\,AD + 2\,BO\,.\,BE + 2\,CO\,.\,CF$.

Dem.—$AC^2 = AO^2 + OC^2 + 2\,AO\,.\,OD$ (II. XII.), $BC^2 = CO^2 + OB^2 + 2\,CO\,.\,OF$, and $AB^2 = AO^2 + OB^2 + 2\,BO\,.\,OE$. Adding, we get $AB^2 + BC^2 + CA^2 = (2\,AO^2 + 2\,AO\,.\,OD) + (2\,OB^2 + 2\,OB\,.\,OE) + (2\,CO^2 + 2\,CO\,.\,OF) = 2\,AO\,(AO + OD) + 2\,BO\,(BO + OE) + 2\,CO\,(CO + OF) = 2\,AO\,.\,AD + 2\,BO\,.\,BE + 2\,CO\,.\,CF$.

(2) Describe a ⊙ about ABC, and from its centre Q let fall $\perp^s$ QG, QH, QJ on the sides, and join AQ. Now (III.) AJ = BJ: $\therefore$ $AB^2 = 4\,AJ^2 = 4\,AQ^2 - 4\,QJ^2$; but AQ = R, and 2 QJ = OC

("Sequel," Book I., Prop. xii., Cor. 3); $\therefore AB^2 = 4R^2 - OC^2$. Similarly, $BC^2 = 4R^2 - OA^2$, and $CA^2 = 4R^2 - OB^2$. Hence $AB^2 + BC^2 + CA^2 = 12R^2 - (OA^2 + OB^2 + OC^2)$.

55. **Dem.**—Join the centres O, O'. Produce DC, and let it meet the circles again in the points G, H. Join OG, O'H.



Now the $\angle$ DCO' = OCG (I. xv.); but OCG = OGC; $\therefore$ DCO' = OGC, and O'DC = ODG (hyp.); $\therefore$ the Δs ODG, O'DC are equiangular; hence (xxxv., Cor. 3) OG . CD = O'C . DG. Again, the $\angle$s O'HD, O'HC are equal to two right $\angle$s; and the $\angle$s OCD, O'CD are equal to two right $\angle$s; and O'CD = O'HC; $\therefore$ the $\angle$ O'HD = OCD, and (hyp.) O'DH = ODC; $\therefore$ the Δs O'HD, OCD are equiangular; hence (xxxv., Cor. 3) O'H.CD = DH . OC. Multiplying these results, we get $CD^2 = DH \cdot DG$. Now $DG \cdot DC = DE^2$ (xxxvi.), and $DH \cdot DC = DF^2$; $\therefore DG \cdot DH \cdot DC^2 = DE^2 \cdot DF^2$; $\therefore DC^4 = DE^2 \cdot DF^2$; $\therefore DC^2 = DE \cdot DF$.

BOOK IV.

PROPOSITION IV.

1. **Dem.**—CF = CD, OC common, and the base OF = OD; hence (I. VIII.) the ∠ OCF = OCD. (Fig., Prop. IV.).

2. **Dem.**—BD = BE, CD = CF, AE = AF (III. XVII.); ∴ CB + AE = $\frac{1}{2}$(AB + BC + CA) = s; that is, a + AE = s; ∴ AE = ($s - a$). In like manner BD = ($s - b$), and CF = ($s - c$). (Fig., Prop. IV.)

3. **Dem.**—From O′ let fall ⊥ˢ O′F, O′G, O′H on the sides AB, BC, CA of the △ ABC. Now, because the ∠ O′CG = O′CH, and the ∠ O′GC = O′HC, and the side O′C common, ∴ (I. XXVI.) O′G = O′H. Similarly, O′G = O′F; ∴ O′F, O′G, O′H are equal, and the ⊙ with O′ as centre, and O′F as radius, will pass through G and H.

4. Let D, E be the points in which AC, CB produced touch the ⊙ whose centre is O‴. It is required to prove that BE = ($s - a$).

Dem.—Let J be the point of contact of AB and O‴. Now it may be proved, as in Ex. 2, that CB + BJ = s; that is, CB + BE = s; but CB = a; hence BE = ($s - a$).

5. (1) It is required to prove that the points O, O‴, A, B are concyclic.

Dem.—Let E be the point in which BC produced touches O‴. Now since the ∠ˢ ABC, ABE are bisected, the ∠ OBO‴ is equal to half the sum of the ∠ˢ ABC, ABE, and is therefore a right angle. Similarly, OAO‴ is a right angle; ∴ the ∠ˢ OAO‴, OBG‴ are together equal to two right angles. Hence (III. XXII.) the points O, O″, A, B are concyclic.

(2) It can be shown as in (1) that the ∠ˢ O′AO″, O′BO″ are right ∠ˢ. Hence (III. XXII., Cor. 1) the points O′, B, A, O″ are concyclic.

6. It is required to prove that O is the orthocentre of the △ O′O″O‴.

Dem.—Because the $\angle$ O″BO‴ is right, O″B is the perpendicular from O″ on O′OO″. Similarly, O′A, O‴C are the perpendiculars from O′, O‴ on O″O‴, O′O″. Hence the point O is the orthocentre of the $\triangle$ O′O″O‴.

7. See Book I., Miscellaneous Ex. 36.

8. **Dem.**—It can be shown, as in Ex. 5, that the four points O, A, O‴, B are concyclic; hence (III. xxi.) the $\angle$ AO‴O = ABO; but ABO = CBO; ∴ CBO = AO‴C, and the $\angle$ ACO‴ = BCO, since ACB is bisected; hence (I. xxxii., Cor 2) the $\triangle$s BOC, ACO‴ are equiangular; ∴ (III. xxxv., Cor. 3) CO . CO‴ = BC . AC = ab. In like manner AO . AO′ = bc, and BO . BO″ = ca.

10. **Dem.**—From O′ let fall $\perp$s r' on AB and AC produced, and on BC join O′A, O′B, O′C. Now $br' = 2 \triangle$ ACO′, $cr' = 2$ ABO′; ∴ $r'(b + c)$ = twice the quadrilateral ACO′B, and $ar' = 2$ BO′C; ∴ $r'(b + c - a) = 2$ ABC; but $(a + b + c) = 2s$; ∴ $(b + c - a) = 2(s - a)$; ∴ $2r'(s - a) = 2$ ABC. Hence $r'(s - a)$ = area of the $\triangle$ ABC.

11. From O, O′ let fall $\perp$s OK, O′H on AC. It is required to prove that OK . O′H = $(s - b)(s - c)$; that is, $rr' = (s - b)(s - c)$.

Dem.—The line AH = s (Ex. 4), and CH, CK are $(s - b)$ and $(s - c)$ (Exs. 4 and 2). Now the $\angle$ OCO′ is right (Ex. 5), ∴ the $\angle$s OCK, O′CH are together equal to a right $\angle$; and since the $\angle$ O′HC is right, the $\angle$s HO′C, HCO′ make together a right $\angle$; ∴ the $\angle$ HO′C = OCK, and the $\angle$ O′HC = OKC, each being right; ∴ the $\triangle$s O′HC, OKC are equiangular. Hence (III. xxxv., Cor. 3) OK . O′H = $(s - b)(s - c)$; that is, $rr' = (s - b)(s - c)$.

12. **Dem.**—Area of $\triangle$ ABC = rs (Ex. 9), and $r'(s - a)$ = area of ABC (Ex. 10); ∴ $r . r' . s . s - a$ = square of area of ABC; but $rr' = s - b . s - c$ (Ex. 11). Hence square of area of ABC $= s . s - a . s - b . s - c$.

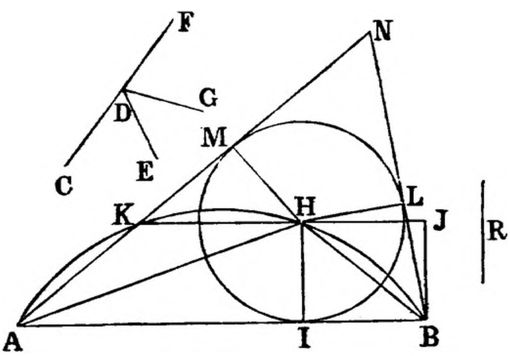
13. **Dem.**—Let the area of ABC be denoted by Δ. Now $rs = \Delta$ (Ex. 9), and $r' . s - a = \Delta$ (Ex. 10). Similarly $r'' . s - b = \Delta$, and $r''' . s - c = \Delta$; hence $(r . r' . r'' . r''')(s . s - a . s - b . s - c) = \Delta^2$; but $(s . s - a . s - b . s - c) = \Delta^2$ (Ex. 12). Therefore $r . r' . r'' . r''' = \Delta^2$.

14. **Dem.**—From O‴ let fall $\perp$s O‴D, O‴D′ on CB, CA. Now in the $\triangle$ O‴D′C the $\angle$ O‴D′C is right, and the $\angle$ D′CO‴ is half a right $\angle$; ∴ the $\angle$ CO‴D′ is half right; ∴ (I. vi.) D′O‴ = D′C; but D′O‴ = r'' and D′C = s (Ex. 4); ∴ $r''' = s$. Similarly it can be shown, if we let fall $\perp$s OE, OE′ from O on CB, CA, that E′C

= E′O; but E′O = r, and E′C = $(s - c)$ (Ex. 2); ∴ $r - (s - c)$. In like manner $r' = (s - b)$, and $r'' = (s - a)$.

15. (1) Let AB be the base, CDE the vertical ∠, and R the radius of the inscribed circle. It is required to construct the triangle.

Sol.—Produce CD to F, and bisect the ∠ EDF by DG. On AB describe a segment of a ⊙ containing an ∠ = CDG. Erect BJ ⊥ to AB and = R. Through J draw JH ∥ to AB, and cut-



ting the ⊙ in H. Join AH, BH, and let fall a ⊥ HI on AB. At the points A, B, in the lines AH, BH, make the ∠ˢ HAK, HBL respectively equal to the ∠ˢ HAB, HBA, and produce AK, BL to meet in N. ANB is the required triangle.

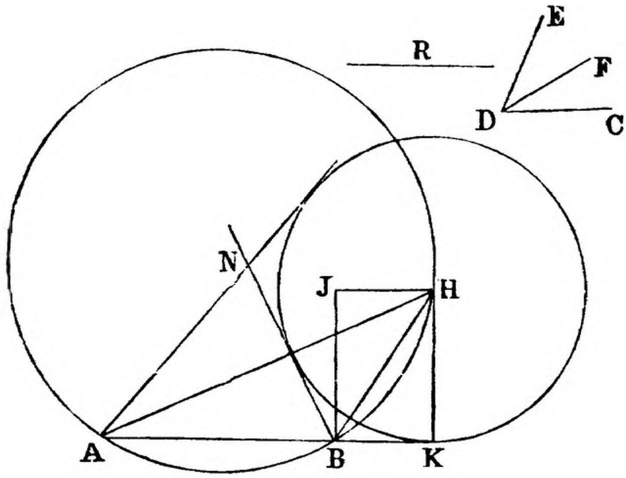
Dem.—From H let fall ⊥ˢ HM, HL on AN, BN. Now in the Δˢ HIB, HLB we have the ∠ˢ HIB, HLB = HLB, HBL, and the side HB common; ∴ (I. xxvi.) HI = HL. Similarly HI = HM; hence the ⊙ with H as centre, and HI as radius, will pass through L and M, and its radius = R, for HI = BJ = R.

Again, the ∠ˢ of the Δ HAB are equal to two right ∠ˢ, and the ∠ˢ CDG, FDG are equal to two right ∠ˢ; but the ∠ AHB = CDG, ∴ the ∠ FDG = HAB + HBA; and because the ∠ˢ of the Δ ANB are two right ∠ˢ, ∴ the ∠ˢ of ANB are equal to the ∠ˢ CDG, FDG; but the ∠ˢ NAB + NBA = 2 (HAB + HBA) = 2 FDG = FDE. Hence the remaining ∠ ANB = CDE.

(2) Let AB be the base, CDE the vertical ∠, and R the radius of the escribed ⊙ which touches the base and one of the sides produced.

Sol.—Bisect the ∠ CDE by DF, and on AB describe a segment containing an ∠ = CDF. Erect BJ ⊥ to AB and = R.

Through J draw JH ∥ to AB, and from H, where it meets the ⊙, let fall a ⊥ HK on AB produced. With H as centre, and HK

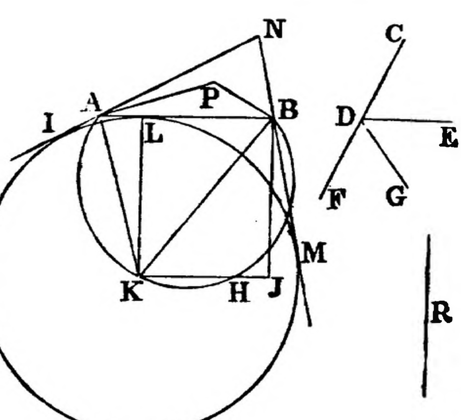


as radius, describe a ⊙. From A, B draw tangents to this ⊙, meeting in N. ANB is the required triangle.

Dem.—Join AH, BH. Now HK = JB = R, and because H is the centre of the escribed ⊙ of the △ ANB, AH, BH are the bisectors of the internal ∠ NAB and the external ∠ NBK (I. xxxii., Ex. 12), the ∠ AHB = $\frac{1}{2}$ANB; but AHB = $\frac{1}{2}$CDE. Hence ANB = CDE.

(3) Let AB be the base, CDE the vertical ∠, and R the radius of the escribed ⊙ which touches the base externally and the sides produced.

Sol.—Produce CD to F, and bisect the ∠ EDF by DG. On



AB describe a segment of a ⊙ containing an ∠ = EDG. Erect

BJ ⊥ to AB, and make it equal to R. Through J draw JK ∥ to AB, and cutting the ⊙ in K. From K let fall a ⊥ KL on AB. With K as centre, and KL as radius, describe a ⊙. Through A, B draw tangents IN, MN to this ⊙, meeting in N. ANB is the triangle required.

Dem.—Join KA, KB. Since K is the centre of the escribed ⊙ of the △ ABN, the ∠ AKB = ½ (NAB + ABN) (I. xxxii., Ex. 12); but AKB = ½ FDE (const.); ∴ NAB + ABN = FDE; hence the ∠ ANB = CDE, and LK, the radius of the escribed ⊙, = BJ = R.

PROPOSITION V.

2. **Dem.**—Because each of the ∠s APB, AQB is right, AQPB is a cyclic quadrilateral, and AP, BQ are chords in the ⊙; hence (III. xxxv.) OA . OP = OB . OQ. Similarly OB . OQ = OC . OR. (Diagram 2, Ex. 1.)

3. **Dem.**—The ∠ AOF = DOC (I. xv.), and AFO = CDO, each being right; FAO = OCD; but OCD = GAF (III. xxi.): ∴ FAO = GAF, and AFO = AFG, each being right, and AF common. Hence (I. xxvi.) OF = GF. (Diagram, Ex. 1.)

5. **Dem.**—In the △ O′O″O‴ the lines O′A, O″B, O‴C are ⊥s from O′, O″, O‴ on O″O‴, O‴O′, O′O″ (iv., Ex. 6), and the points A, B, C are the feet of these ⊥s; hence (Ex. 4), the ⊙ about ABC is the nine-points ⊙ of the △ O′O″O‴. In like manner it is the nine-points ⊙ of the △s OO′O″, OO″O‴, OO′O‴. (Diagram, Ex. 3, Prop. iv.)

6. **Dem.**—Because the lines IF, IH, IK are equal (Ex. 4), and the ∠ KFH is right, HK is the diameter of the ⊙ about the △ KFH; ∴ IK, IH are in one straight line; and since KH is ∥ to CP, and CK to PH, PCKH is a parallelogram; ∴ CK = PH; but CO = 2 CK, ∴ CO = 2 PH. (Diagram, Ex. 4.)

7. **Dem.**—IF = ½ PG. This is proved in Ex. 4.

PROPOSITION X.

1. **Dem.**—The ∠ ACD = CBD + CDB (I. xxxii.); but CBD = 2 CAD (x.), and CDB = CAD. Hence the ∠ ACD = 3 CAD.

2. **Dem.**—The ∠s of the △ ABD are equal to two right ∠s; but each of the ∠s ABD, ADB is equal to 2 BAD; hence the ∠ BAD is $\frac{1}{5}$ of two right ∠s; that is, $\frac{1}{10}$ of four right ∠s; ∴ the arc BD is $\frac{1}{10}$ of the whole circumference. Hence the line BD is a side of a regular decagon.

3. **Dem.**—Let A be the centre. Join AB, AD, AE, AF, and join BF, cutting AD in G. Now since BD is a side of a regular inscribed decagon, ABD is an isosceles △, having each of its base ∠s double of the vertical ∠ (Ex. 2), the ∠ BAD is $\frac{1}{5}$ of two right ∠s, ∴ the ∠ BAF is $\frac{3}{5}$ of two right ∠s; hence the ∠ AFB is $\frac{1}{5}$ of two right ∠s, ∴ the ∠ AGF is $\frac{2}{5}$ of two right ∠s; ∴ AF = GF, that is, BF − BG = R. Now the ∠ DBG is $\frac{1}{5}$ of two right ∠s, and BDG is $\frac{2}{5}$, ∴ BGD is $\frac{2}{5}$, ∴ BG = BD. Hence BF − BD = R.

4. **Dem.**—Because ACDE is a cyclic quadrilateral, the ∠s ACD, AED are together equal to two right ∠s (III. xxii.); and the ∠s ACD, BCD are together = to two right ∠s, ∴ the ∠ AED = BCD; that is, AED = CBD; but AED = ADE, and CBD = ADB, ∴ ADE = ADB, and AD common. Hence (I. xxvi.) DE = DB.

Again, the ∠ ACE = ADE (III. xxi.), and the ∠ CDA = CEA; but (x.) CDA = CAD = DAE; ∴ CEA = DAE, and the side AE = AD. Hence (I. xxvi.) the △s ACE, ADE are congruent.

5. **Dem.**—Let O be the centre of the ⊙ ACD. Join OA, OC. Now (Ex. 4) AEC is an isosceles △, having each base ∠ double of the vertical ∠; and since the ∠s of the △ AEC are together equal to two right ∠s, the ∠ AEC is $\frac{1}{5}$ of two right ∠s; hence (III. xx.) the ∠ AOC is $\frac{2}{5}$ of two right ∠s; that is, $\frac{1}{5}$ of four right ∠s. Hence AC is the side of a regular pentagon.

PROPOSITION XI.

1. Let ABCDE be a regular pentagon inscribed in a ⊙, and let its diagonals CE, AD intersect in A′; BD, CE in B′; CA, BD in C′; AC, BE in D′; and BE, AD in E′. It is required to prove that A′B′C′D′E′ is a regular pentagon.

Dem.—Because the arc AE = BC (XI.), the ∠ ECA = BAC, ∴ CE is ∥ to AB; hence (I. XXIX.) the ∠s EB′B, B′BA, are together equal to two right ∠s; for the same reason the ∠s CA′A, A′AB are equal to two right ∠s; but the ∠ DBA = DAB; hence the ∠ A′B′B = B′A′A. In like manner the ∠s at C′, D′, E′ are equal. Hence the figure A′B′C′D′E′ is equiangular.

Again, because the arc BC = DE, the ∠ BDC = DCE; ∴ the side B′C = B′D, and (I. XV.) the ∠ CB′C′ = A′B′D; and the ∠ B′C′C = B′A′D, because they are the supplements of the equal ∠s B′C′D′, B′A′E′; hence the side C′B′ = A′B′. Similarly, the other sides of A′B′C′D′E′ are equal. Hence it is a regular pentagon.

2. Produce AE, CD to meet in A′; ED, BC in B′; DC, AB in C′; CB, EA in D′; BA, DE in E′. Join A′B′, B′C′, &c. It is required to prove that A′B′C′D′E′ is a regular pentagon.

Dem.—In the △s ABD′, CBC′, the ∠ ABD′ = CBC′, and the ∠ D′AB = BCC′, being the supplements of equal ∠s, and the side AB = CB; hence (I. XXVI.) BD′ = BC′, and the ∠ AD′B = BC′C. Similarly, AD′ = AE′; EE′ = EA′; DA′ = DB′; and CB′ = CC′. Again, because the ∠ ABC = EAB, the ∠ D′BA = D′AB; ∴ D′A = D′B. Now in the △s D′AE′, D′BC′, we have the sides AD′, AE′ = BD′, BC′, and the contained ∠s equal; hence the base D′E′ = D′C′. In like manner the other sides are equal. Hence the figure is equilateral. Again, we proved the ∠ BD′C′ = BC′D′, and the ∠ AD′B = BC′C; and the ∠ AD′E is = CC′B′, since the △s AD′E′, CC′B′ are equal in every respect. Hence the ∠ E′D′C′ = D′C′B′. In like manner the other ∠s are equal. Hence the pentagon A′B′C′D′E′ is regular.

3. Let AD, BE, two consecutive diagonals of a regular pentagon, intersect in E′. It is required to prove that EB . EE′ = E′B².

Dem.—Join CE, and describe a ⊙ about the △ AE′B.
Now because DE = BC, the ∠ DCE = BEC; ∴ DC is ∥ to BE.

Similarly, BC is ∥ to AD; hence (I. xxxiv.) DC = BE′; but DC = AB (hyp.): ∴ AB = BE′. Again, because AE = DE, the ∠ ABE = EAD, and hence (III. xxxii.) AE is a tangent to the ⊙ ABE′; ∴ (III. xxxvi.) EB . EE′ = AE^2 = AB^2 = $E'B^2$. Hence BE is cut in extreme and mean ratio in E′.

4. Let AB be a side of a regular pentagon. It is required to construct it.

Sol.—Erect BC ⊥ to AB, and make it equal to $\frac{1}{2}$ AB. Join AC, and produce it to D, so that CD = CB. On AB describe an isosceles △ ABE, having each of its equal sides equal to AD. About the △ ABE describe a ⊙. Bisect the ∠s BAE, ABE by the lines AF, BG, meeting the circumference in F and G. Join AG, GE, EF, BF. ABFEG is the required pentagon.

Dem.—From AC cut off CH = CB or CD. Now DA . AH + CH^2 = AC^2 (II. vi.); but CH^2 = BC^2, and AC^2 = AB^2 + BC^2; ∴ DA . AH = AB^2 = DH^2; ∴ AD is divided in extreme and mean ratio in H. Therefore, since AE = AD, if we divide AE in extreme and mean ratio, the greater segment would be equal to AB, and hence (x.) AEB is an isosceles △, having each base ∠ double the vertical ∠; but the base ∠s are bisected by the lines AF, BG; ∴ the ∠s EAF, FAB, ABG, GBE, AEB are equal; ∴ the chords EF, BF, AG, EG, AB are equal. Hence ABFEG is a regular pentagon.

5. Let ABC be a right ∠. It is required to divide it into five equal parts.

Sol.—Draw BD, making the ∠ ABD equal to the vertical ∠ of an isosceles △ having each of its base ∠s double the vertical ∠. Bisect the ∠ ABD by BE; each of the ∠s ABE, DBE is $\frac{1}{5}$ of a right ∠. Draw BF, BG, making the ∠s DBF, FBG each equal to EBD. Then the ∠ ABC is divided into five equal parts by the lines BE, BD, BF, BG.

PROPOSITION XV.

1. (1) Let ABCDEF be the hexagon. Join, AC, AE, CE. It is required to prove that the area of the hexagon is double the area of the △ ACE.

Dem.—Let the diagonals of the hexagon intersect in O. Now

the △ˢ OCD, OED are equilateral, and hence OCDE is a lozenge, and CE is its diagonal; ∴ OCDE = 2 OCE. Similarly OACB = 2 OAC, and OAFE = 2 OAC. Hence ABCDEF = 2 ACE.

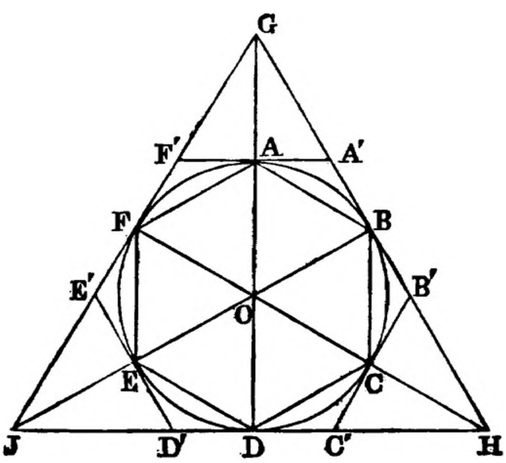
(2) See Book I., Prop. I., Ex. 4.

2. Let AB be the diameter, and O the centre. Produce AB to C, so that BC = BO. From C draw tangents CD, CE to the ⊙, and join DE. It is required to prove that the △ CDE is equilateral.

Dem.—Join OD, OE, BD, BE. Now (III. XVIII.) the ∠ CDO is right; ∴ (Book I., Prop. XII., Ex. 2) the lines BD, BO, BC are equal; but OB = FO; ∴ the △ ODB is equilateral; and because each of the ∠ˢ CDO, CEO is right, CDOE is a cyclic quadrilateral, ∴ the ∠ˢ DOE, DCE are together equal to two right ∠ˢ; but each of the ∠ˢ DOB, BOE is an ∠ of an equilateral △, ∴ DCE is an ∠ of an equilateral △; and because CD = CE, the △ CDE is equilateral.

3. (1) Let ABCDEF be the hexagon, and GHJ the equilateral △. It is required to prove that the area of the △ is double the area of the hexagon.

Dem.—Let the diagonals of the hexagon intersect in O. Join



AG, CH, EJ. Now, because AB = AF, AG common, and the base GB = GF, ∴ (I. VIII.) the ∠ BAG = FAG, and the ∠ OAB = OAF; ∴ the ∠ˢ FAG, OAF are together equal to two right ∠ˢ; hence (I. XIV.) OA and AG are in the same straight line.

Again (III. XVIII.), the ∠ OFG is right, ∴ the ∠ˢ FOG, FGO make one right ∠; but the ∠ AFO = FOA; ∴ the ∠ AFG

= AGF; ∴ AF = AG; but AO = AF; ∴ AO = AG; hence (I. xxxvii.) the △ AFO = AFG; ∴ the △ OFG = 2 OFA. Similarly, OBG = 2 OBA; ∴ OFGB = 2 OFAB. In like manner OBHD = 2 OBCD, and OFJD = 2 OFED. Hence the △ GHJ = 2 ABCDEF.

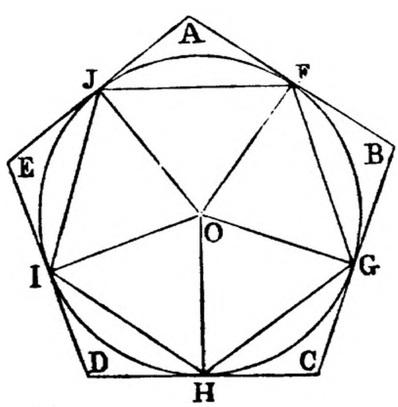
(2) Let A′B′C′D′E′F′ be the circumscribed hexagon. It is required to prove that the area of ABCDEF is three-fourths the area of A′B′C′D′E′F′.

Dem.—Because each of the ∠s F′AO, F′FO is right (III. xviii.), the ∠s AF′F, AOF are together equal to two right ∠s, and the ∠s AF′F, AF′G are together equal to two right ∠s; hence the ∠ AF′G = AOF; ∴ AF′G is an ∠ of an equilateral △. In like manner AA′G is an ∠ of an equilateral △; ∴ GF′A′ is an equilateral △; and because GA is ⊥, it bisects the base; ∴ AF′ = AA′; ∴ A′F′ or GF′ = 2 AF′ = 2 FF′; hence the △ F′GA = 2 FF′A; ∴ FGA = 3 FF′A; hence (1) AOF = 3 FF′A; ∴ AOF = $\frac{3}{4}$ OFF′A. In like manner AOB = $\frac{3}{4}$ OAA′B, &c. Hence ABCDEF = $\frac{3}{4}$ A′B′C′D′E′F′.

Exercises on Book IV.

1. (1) Let ABCDE be a regular polygon circumscribing a ⊙. It is required to prove that the corresponding inscribed polygon is regular.

Dem.—Let O be the centre. Join OF, OG, OH, OI, OJ.



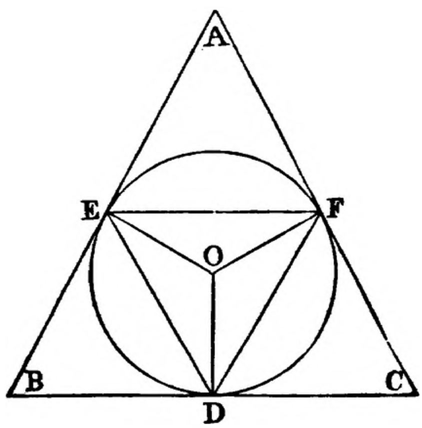
Now (III. xviii.) the ∠s OHD, OID are right, ∴ the

$\angle$s IDH, IOH are together equal to two right $\angle$s. In like manner the $\angle$s GCH, GOH are together equal to two right $\angle$s; but IDH = GCH (hyp.); ∴ the $\angle$ IOH = GOH. In the same way it can be shown that all the $\angle$s at O are equal. Hence the arcs are all equal, and therefore the five chords FG, GH, HI, IJ, JF are all equal.

(2) Proved as in Book IV., Prop. XII.

2. Let the $\triangle$ ABC be isosceles. It is required to prove that the $\triangle$ DEF is isosceles.

Dem.—Let O be the centre. Join OD, OE, OF. Now the $\angle$s ODB, OEB are right (III. XVIII.), ∴ the $\angle$s EBD, EOD are together equal to two right $\angle$s. Similarly the $\angle$s



FCD, FOD are together equal to two right $\angle$s; but the $\angle$ EBD = FCD (hyp.); ∴ the $\angle$ EOD = FOD; ∴ the arc ED = FD; ∴ the chord ED = FD. And hence the $\triangle$ DEF is isosceles.

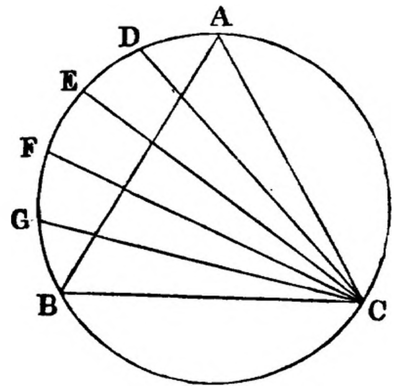
3. Let the $\angle$ BAC = EDF. It is required to prove that both $\triangle$s are equilateral.

Dem.—Because the $\triangle$s are isosceles, and the $\angle$ BAC = EDF, their remaining $\angle$s are equal; ∴ the $\angle$ ABC = EFD; but EFD = EDB (III. XXXII.); ∴ EBD = EDB, and EDB = BED; ∴ EBD is an $\angle$ of an equilateral $\triangle$. Similarly FCD is an $\angle$ of an equilateral $\triangle$. Hence ABC and DEF are equilateral triangles.

4. Let ACB be an $\angle$ of an equilateral $\triangle$. It is required to divide it into five equal parts.

Sol.—Describe a ⊙ about the $\triangle$ ABC, and in it inscribe a regular polygon of fifteen sides (XVI.); then five of those sides

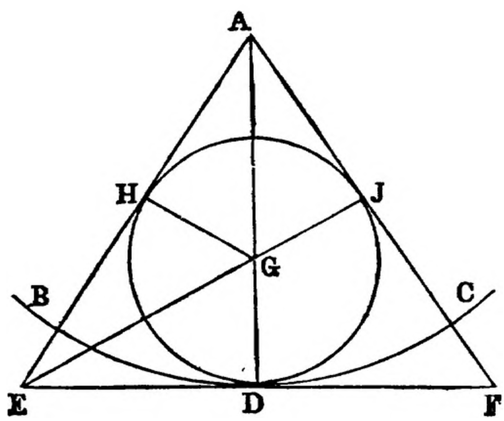
will be in the arc AB. Let D, E, F, G be the points of division.



Join CD, CE, CF, CG. Now since the arcs AD, DE, EF, FG, GB are equal, the ∠s ACD, DCE, ECF, FCG, GCB are equal.

5. Let ABC be a sector of a given circle. It is required to inscribe a circle in it.

Sol.—Bisect the ∠ BAC by AD, meeting the arc BC in D. Through D draw EF a tangent to the sector. Produce AB, AC

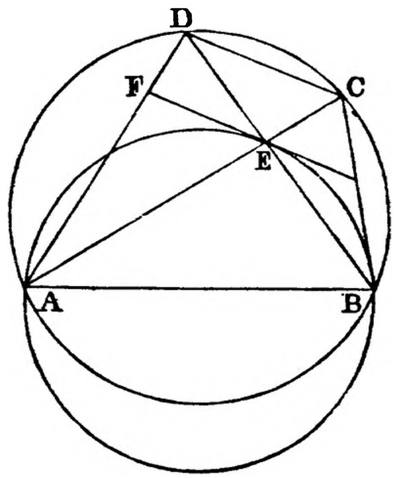


to meet EF. Bisect the ∠ AEF by EG, meeting AD in G. G is the centre of the required circle.

Dem.—From G let fall ⊥s GH, GJ on AE, AF. Now (III. xviii.) the ∠ EDG is right, and the ∠ EHG is right (const.), and the ∠ DEG = HEG, and EG common; ∴ (I. xxvi.) GD = GH. Similarly GH = GJ. Hence the ⊙, with G as centre and GD as radius, will pass through H and J.

6. **Dem.**—Describe a ⊙ about ABC, and through A draw AF touching this ⊙. Now (III. XXXII.) the ∠ FAC = ABC; but ABC = ADE (I. XXIX.); ∴ FAC = ADE; ∴ the ⊙ about ADE will touch AF in A. Hence the circles touch each other in A.

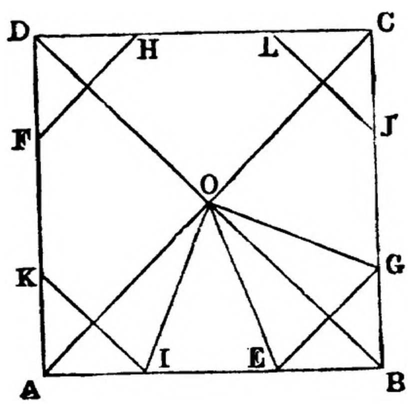
7. **Dem.**—Let EF be the tangent at E to the circle about ABE. Now the ∠ FEA = EBA (III. XXXII.); but EBA = DCA (III. XXI.). Hence the ∠ FEA = DCA, and therefore the lines EF, CD are parallel.



8. Let ABCD be a given square. It is required to describe a regular octagon in it.

Sol.—Draw the diagonals AC, BD, intersecting in O. Cut off AE, AF = AO; BI, BJ = BO; CG, CH = CO; DK, DL = DO. Join EG, JL, HF, KI. EGJLHFKI is the octagon required.

Dem.—Join OE, OG, OI. Now, because AE = AO, and the

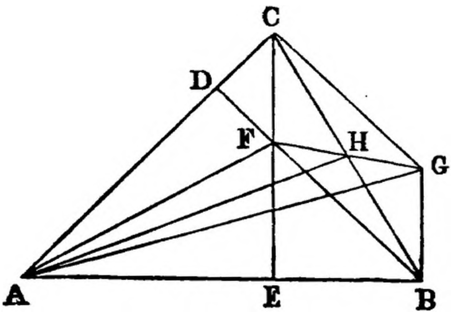


∠ EAO is half a right ∠, ∴ each of the ∠s AEO, AOE is three-

fourths of a right ∠, and the ∠ AOB is right; ∴ EOB is one-fourth of a right ∠. Similarly, each of the ∠s GOB, AOI is one-fourth of a right ∠; hence EOI is half a right ∠, and we have seen that AEO is three-fourths of a right ∠; ∴ EIO is three-fourths of a right ∠; ∴ OI = OE. And because the ∠ EOB = GOB, and EBO = GBO, and the side BO common, OG = OE = OI. Now OG = OI, and OE common, and the ∠ GOE = IOE; ∴ the bases EG, EI are equal. In like manner all the sides are equal. Again, because BE = BG, the ∠ BEG = BGE; ∴ each is half a right ∠; each of the ∠s GEI, EGJ is three halves of a right ∠. In like manner all the ∠s are equal. Hence the octagon is regular.

9. Let AB, AC be two given lines, and BC a line of given length sliding between them. From B, C ⊥s BD, CE are let fall on AC, AB, intersecting in F. It is required to find the locus of F.

Sol.—At B, C erect ⊥s BG, CG to AB, AC. Join FG, cutting BC in H. Join AF, AG, AH. Now, because CE and GB are perpendicular to AB, and CG, BD to AC, CGBF is a parallelogram; hence (I. xxxiv., Ex. 1) BH = CH, and FH = GH. Again, since BC is a line of given length sliding between two fixed lines, AB, AC, and BG, CG are perpendiculars at its ex-

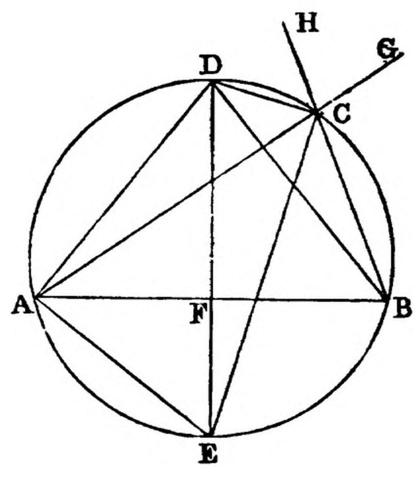


tremities; ∴ (III. xxviii., Ex. 2) the locus of G is a ⊙, having A as centre, and AG as radius; hence AG is a given line, and (I. xlvii.) $AC^2 + CG^2 = AG^2$, and $AB^2 + BG^2 = AG^2$; ∴ $AC^2 + CG^2 + AB^2 + BG^2$ is given; but (II. x., Ex. 2) $BG^2 + CG^2 = 2\,CH^2 + 2\,HG^2$, and $AB^2 + AC^2 = 2\,CH^2 + 2\,AH^2$; ∴ $4\,CH^2 + 2\,AH^2 + 2\,GH^2$ is given; but $4\,CH^2 = CB^2$; ∴ $4\,CH^2$ is given, and ∴ $2\,AH^2 + 2\,GH^2$ is given; ∴ $AF^2 + AG^2$ is given;

but AG^2 is given, $\therefore$ AF is given; hence AF is a line of given length; and since A is a fixed point, the locus of F is a ⊙ having A as centre, and AF as radius.

10. Let ABC be the triangle. About ABC describe a ⊙. Let DF be a ⊥ at the middle point of AB. Produce DF to meet the circumference in E. Join AD, BD, CD, CE. It is required to prove that CE is the internal, and CD the external bisector of the ∠ ACB.

Dem.—Produce BC to H, and join AE, BE. Now, because AF = BF and FE common, and the ∠ AFE = BFE, the base AE is equal to BE; $\therefore$ the arc AE = BE; hence the ∠ ACE = BCE. Therefore CE is the internal bisector of the ∠ ACB. Again (I. IV.), AD = BD, and the ∠ FAD = FBD;

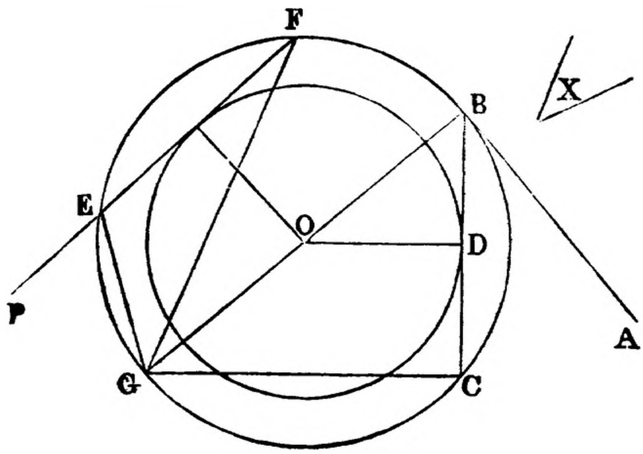


and because ABCD is a cyclic quadrilateral, the ∠ˢ BAD and BCD are together equal to two right ∠ˢ, and the ∠ˢ BCD, DCH are together equal to two right ∠ˢ; $\therefore$ the ∠ BAD = DCH, and (III. XXI.) the ∠ ACD = ABD, and ABD = BAD; hence ACD = DCH. Therefore CD is the external bisector of the ∠ ACB.

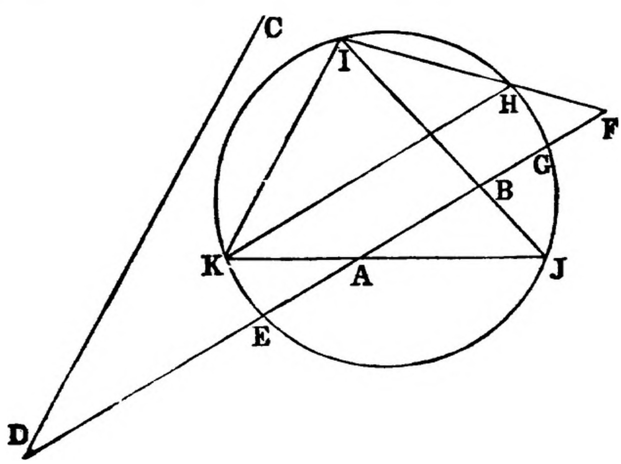
11. **Sol.**—Draw any tangent AB to the ⊙. At the point B make the ∠ ABC = X. From the centre O draw OD ⊥ to BC; and with O as centre, and OD as radius, describe a ⊙. Through P draw a tangent to this ⊙, cutting BCE in E and F. PEF is the line required.

Dem.—Take any point G in BCE, and join BG, CG, EG, FG.

Now (III. xiv.) EF = BC, ∴ the ∠ EGF = BGC; but (III. xxxii.) BGC = ABC = X. Hence the ∠ EGF = X.



12. Let IJK be the given ⊙; A, B the given points, and CD the given line. It is required to inscribe a △ in IJK, having



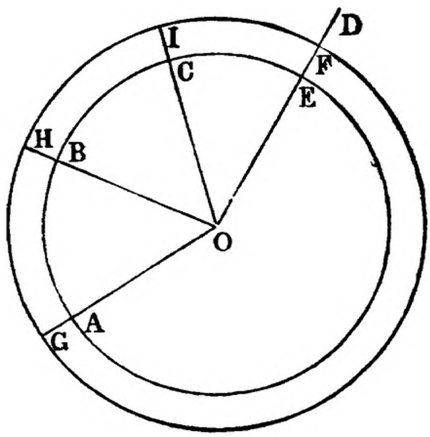
two of its sides passing through A, B, and the third parallel to CD.

Sol.—Join BA, and produce it to meet CD. Produce EB to F, and make AB . BF = EB . BG. Through F draw FHI, cutting the ⊙ in H, I, and subtending an ∠ = CDF (Ex. 11). Join IB, and produce it to meet the ⊙ in J. Join JA, and produce it to meet the ⊙ in K. Join IK. IJK is the required triangle.

Dem.—Join HK. Now, because AB . BF = EB . BG, ∴ AB . BF = JB . BI; hence the points A, J, F, I are concyclic, ∴ the ∠ AJI = AFI; but AJI = KHI (III. xxi.); ∴ AFI

= KHI. Hence AF and KH are parallel; and since the ∠ HKI = CDF, IK is ∥ to CD.

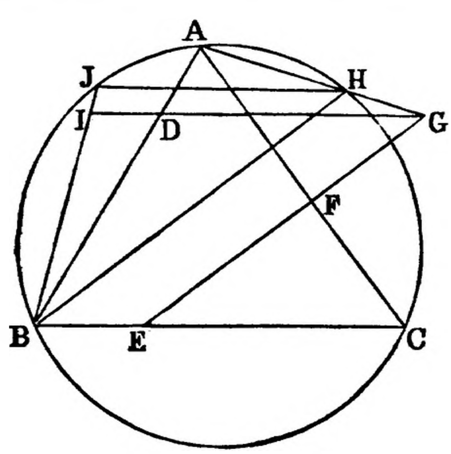
13. Let A, B, C, D be four points, no three of which are collinear. It is required to describe a ⊙ which shall be equidistant from them.



Sol.—Describe a ⊙ passing through A, B, C. Let O be its centre. Join OD, cutting the ⊙ in E. Bisect ED in F. With O as centre, and OF as radius, describe a ⊙ GHI. This is the ⊙ required.

Dem.—Join OA, OB, OC, and produce them to meet the ⊙ GHI. Because OF = OI, and OE = OC; ∴ EF = CI; but EF = DF, ∴ CI = DF. In like manner BH and AG are equal to DF. Hence the ⊙ through G, H, I, F is equally distant from the points A, B, C, D.

14. Let ABC be a given ⊙. It is required to inscribe a △ in



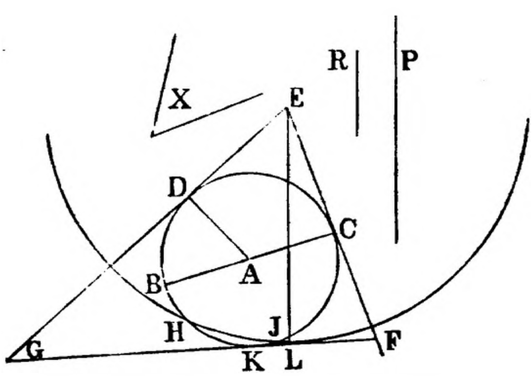
it, whose sides shall pass through three given points D, E, F.

Analysis.—Let ABC be the △. Join EF. Through B draw BH parallel to EF, and meeting the ⊙ in H. Join AH, and produce it to meet EF produced. Join GD. Through H draw HJ parallel to GD, and meeting the ⊙ in J. Join JB, and produce GD to meet JB in I. Now because BH is ‖ to EG, the ∠ CEG = CBH; but CBH = CAH (III. xxi.); ∴ CEG = CAH; hence ECGA is a cyclic quadrilateral; ∴ EF . FG = AF . FC; but AF . FC is given, since F is a given point; ∴ EF . FG is given; but EF is given, ∴ FG is given; ∴ G is a given point.

Again, the ∠ ABJ = AHJ (III. xxi.); but AHJ = AGI (I. xxix.), ∴ ABJ = AGI; hence BIAG is a cyclic quadrilateral; ∴ GD . DI = AD . DB; but AD . DB is given, ∴ GD . DI is given, ∴ DI is given; hence I is a given point. Now the ∠ JHB = IGE (I. xxix., Ex. 8); but the ∠ IGE is given, since the lines EG, IG are given in position; ∴ the ∠ JHB is given. Hence the question reduces to Ex. 11.

15. (1) Let R be the radius of the inscribed circle; X the vertical ∠, and P the perpendicular from the vertical ∠ on the base. It is required to construct the triangle.

Sol.—With any point A as centre, and a radius equal to R, describe a ⊙. Draw BC a diameter, and at the point A in AB



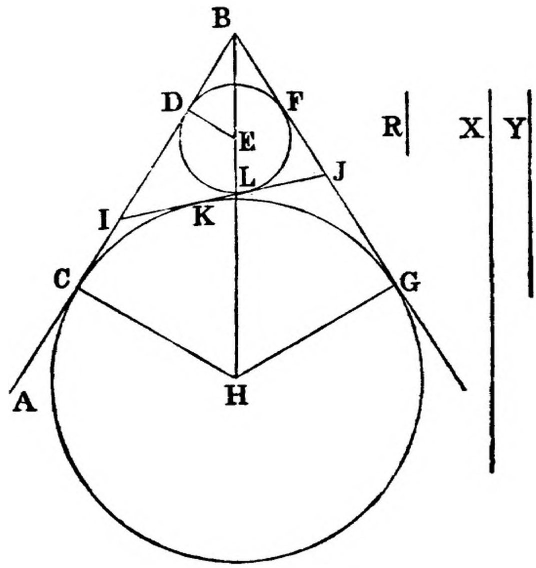
make the ∠ BAD = X. Through C, D draw EF, EG tangents to the ⊙. With E as centre, and a radius equal to P, describe a ⊙. Draw FG a common tangent, touching the ⊙s in K, L. EFG is the required triangle.

Dem.—Join EL. Now the ∠ ELF is right; EL is the ⊥ from the vertical ∠ on the base, and it is equal to P (const.); and AD, the radius of the inscribed ⊙, is equal to R. Again, each of the ∠s ACE, ADE is right (III. xviii.); ∴ the ∠s CAD, CED are

together equal to two right ∠s; and the ∠s CAD, BAD are together equal to two right ∠s; ∴ CED = BAD = X.

(2) Let X be the sum of the sides, Y the base, and R the radius of the inscribed circle.

Sol.—Take a line AB, and from it cut off BC = $\frac{1}{2}$ (X + Y) and CD = Y. Erect DE ⊥ to AB, and make it equal to R. With E as centre, and ED as radius, describe a ⊙. Draw BG, touching this ⊙ at F. Join BE, and produce it. Erect CH ⊥ to AB, and meeting BE produced in H. From H draw HG ⊥ to BG. With H as centre, and HC as radius, describe a ⊙. Draw IJ a common tangent, touching the ⊙s in K and L. BIJ is the required triangle.

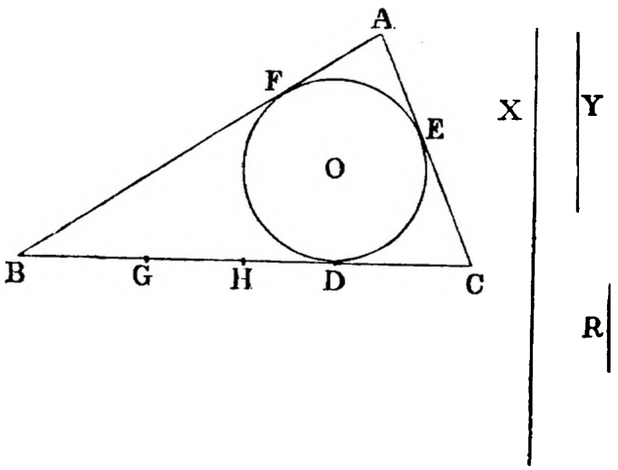


Dem.—ED, the radius of the inscribed ⊙, is equal to R; and (IV., Ex. 2) IJ + BD = $\frac{1}{2}$ (IB + BJ + IJ), and (IV., Ex. 4) BC = $\frac{1}{2}$ (IB + BJ + IJ); hence IJ = CD = Y. Again, BC = $\frac{1}{2}$ (IB + BJ + IJ), and BC = $\frac{1}{2}$ (X + Y) (const.); hence (IB + BJ + IJ) = (X + Y); ∴ (IB + BJ) = X.

(2′) Let X be the base, Y the difference of the sides, and R the radius of the inscribed circle.

Sol.—With any point O as centre, and a radius equal to R, describe a ⊙. In the circumference take a point D. Through D draw a tangent BC. From DB cut off DG = Y. Bisect DG in H, and make BH, CH each equal to $\frac{1}{2}$ X. Through B, C draw AB, AC tangents to the circle. ABC is the triangle required.

Dem.—BC = BH + CH = X; and AB = AF + FB, and AC = AE + EC. Hence AB − AC = FB − EC = BD − CD = BD − BG = DG = Y.

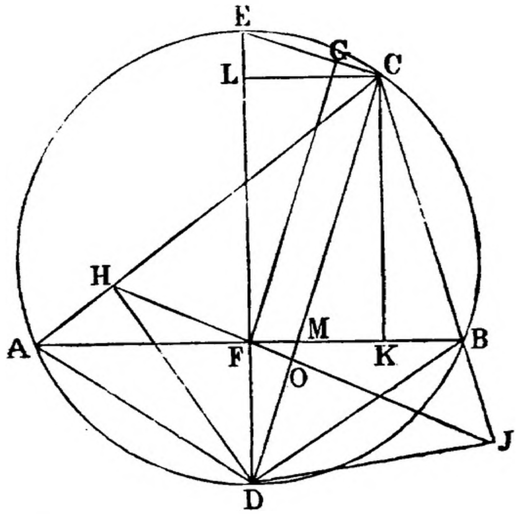


(3) (Diagram, Prop. IV., Ex. 3).—Let O′, O″, O‴ be the centres of the escribed circles. Join them, and let fall ⊥ˢ O′A, O″B, O‴C, on O″O‴, O‴O′, O′O″. Join AB, BC, CA. ABC is the triangle required.

Dem.—Produce AB, AC to F and H. Let O be the point where the ⊥ˢ intersect. Now because each of the ∠ˢ O″CO‴, O′AO″ is right, AOCO″ is a cyclic quadrilateral; ∴ the ∠ ACO″ = AOO″. Similarly the ∠ BCO′ = BOO′; but AOO″ = BOO′, and ACO″ = O′CH, ∴ BCO′ = O′CH; hence CO′ is the external bisector of the ∠ ACB. Similarly, BO′ is the external bisector of the ∠ ABC. Hence O′ is the centre of the escribed ⊙ touching BC externally and the other sides produced. In like manner O″, O‴ are the centres of the other escribed circles.

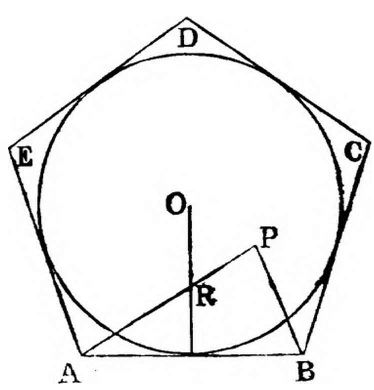
16. (1) **Dem.**—From D let fall ⊥ˢ DH, DJ on AC and CB produced. Join DA, DB, HF, FJ; the points H, F, J are collinear (III. XXII., Ex. 12). Join DC, CE, and through F draw FG parallel to DC. Now because the ∠ ACB is bisected by CD, HC = $\frac{1}{2}$ (AC + CB) (III. XXX., Ex. 4); and since the ∠ DHC is right, DC . CO = HC^2 (I. XLVII., Ex. 1); that is, DC . FG = HC^2. Again (III. XXXI.), the ∠ DCE is right; ∴ EGF is right, and CLD is right; ∴ EGF = CLD, and (I. XXIX.) the ∠ EFG = LDC; ∴ the △ˢ DCL, EFG are equiangular; hence (III. XXXV., Cor. 3) DC . FG = DL . FE; ∴ DL . FE = HC^2.

(2) From C let fall a $\perp$ CK on AB. Now (III. Ex. 17) FM . FK is equal to the square of half the difference of AC and CB; that is, equal to AH^2.



Again, the $\angle$ ELC = DFM, each being right; and because DCE is right, the $\angle$ˢ CED, CDE are together equal to a right $\angle$; and the $\angle$ˢ LEC, LCE are equal to a right $\angle$; ∴ LCE = CDE; hence the $\triangle$ˢ DFM, CLE are equiangular, ∴ (III. xxxv., Cor. 3) DF . LE = LC . FM = FK . FM = AH^2.

17. Let the regular polygon of n sides be a pentagon ABCDE, P a point within it, and p_1, p_2, &c., the $\perp$ˢ from P on the sides.

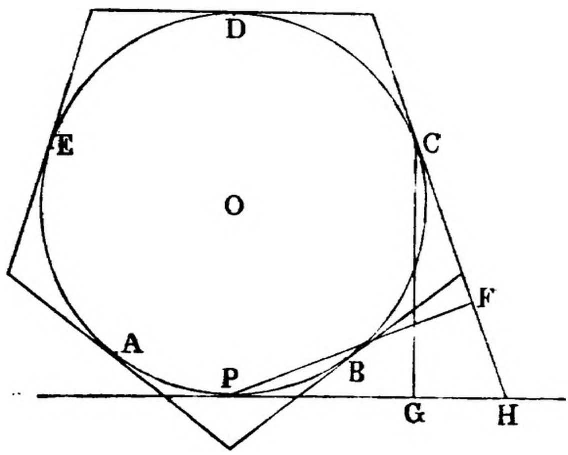


Let O be the centre of the inscribed ⊙, and R its radius. It is required to prove that $(p_1 + p_2 + p_3 + p_4 + p_5) = 5$ R.

Dem.—Join AP, BP, &c., and let the sides be denoted by s. Now sp_1 = twice the $\triangle$ APB; sp_2 = twice the $\triangle$ BPC, &c.; hence

$s(p_1 + p_2 + p_3 + p_4 + p_5) =$ twice the area of the pentagon. Again, $sR =$ twice the Δ AOB, &c., $\therefore 5\,sR =$ twice the area of the pentagon; $\therefore s(p_1 + p_2 + p_3 + p_4 + p_5) = 5\,sR$. Hence $(p_1 + p_2 + p_3 + p_4 + p_5) = 5\,R$. Similarly for a regular polygon of any number of sides.

18. Let A, B, C, D, E be the angular points of a regular polygon of five sides. About ABCDE describe a ⊙; and through



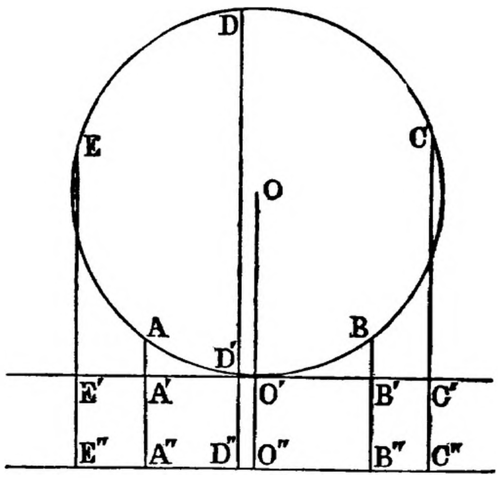
A, B, C, D, E draw tangents to this ⊙. It is required to prove that the sum of the ⊥s from A, B, C, D, E on any line is equal to five times the ⊥ from O, the centre of the ⊙, on the same line.

(1) **Dem.**—Let the line be a tangent at any point P in the circumference. From P, C let fall ⊥s PF, CG on the tangents through C and P. Produce CF to meet PG in H. Now in the Δs CGH, PFH, the ∠ CGH = PFH, and PHC is common, and the side CH = PH; hence (I. xxvi.) CG = PF. Similarly, the ⊥s from A, B, D, E on the tangent at P are equal to the ⊥s from P on the tangents through those points; but (Ex. 17) the sum of the ⊥s from P on the sides of ABCDE is equal to 5 R; hence the sum of the ⊥s from A, B, C, D, E on PH = 5 R; that is, equal to five times the ⊥ from O on PH; and similarly for a regular polygon of any number of sides.

(2) Let the line not touch the circle.

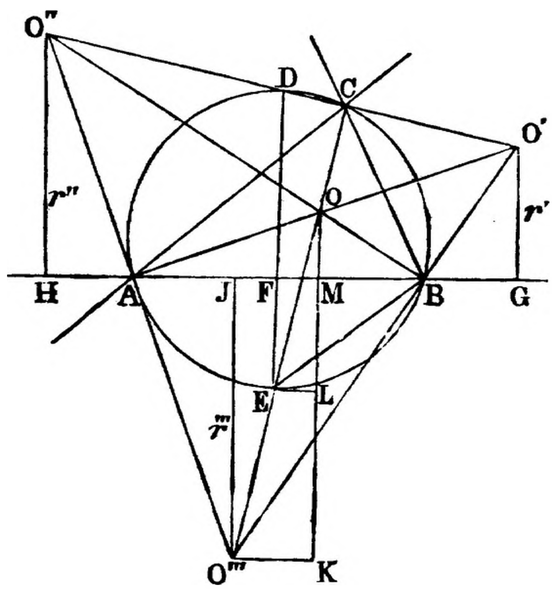
Dem.—From A, B, C, D, E let fall ⊥s AA″, BB″, CC″, DD″, EE″ on the line, and from O let fall OO″. At O′, where OO″ cuts the ⊙, draw a line parallel to C″E″, and let the ⊥s from A, B, C, D, E cut this line in A′, B′, C′, D′, E′. Now (1) AA′ + BB′

$+ CC' + DD' + EE' = 5\,OO'$, and $A'A' + B'B'' + C'C'' + D'D''$



$+ E'E'' = 5\,O'O''$. Hence, by addition, we get $AA'' + BB'' + CC'' + DD'' + EE'' = 5\,OO''$.

19. (1) Let ABC be a Δ inscribed in a $\odot$. Bisect the base AB in F, and erect a $\perp$, meeting the circle in D, E;



then (III. III.) DE is the diameter. Join CD, CE. CD and CE are the external and internal bisectors of the $\angle$ ACB (Ex. 10). Produce AB to G, H. Bisect the $\angle$s CBG, CAH by BO', AO'', meeting CD produced in O', O''. Produce O'B, O''A to meet in

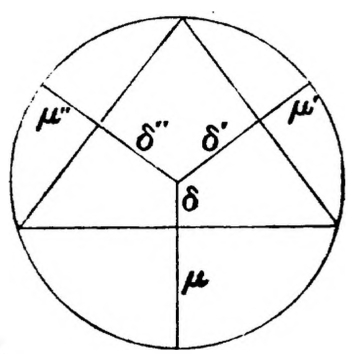
O‴. O′, O″, O‴ are the centres of the escribed ⊙ˢ (IV., Ex. 3). Produce CE. CE produced will pass through O‴ (I. XXVI., Ex. 8). From O‴ let fall a ⊥ O‴J on AB. Join AO′, meeting CE in O. From O draw OK parallel to O″J, and from O‴ and E draw O‴K and EL parallel to AB. From O′, O″ let fall ⊥ˢ O′G (r'), O″H (r'') on GH. Join BE.

Now, since AO′, BO′, CO′ meet in O′, and that BO′, CO′ are two external bisectors, hence (I. XXVI., Ex. 8) AO′ is the internal bisector of the ∠ BAC. Similarly, BO″ is the internal bisector of the ∠ ABC.

Again, AG, BH are each equal to s (IV., Ex. 4); ∴ AH = BG; ∴ HF = GF; hence HG is bisected in F; ∴ (I. XL., Ex. 8) O′G + O″H = 2 DF; that is, $r' + r'' = 2\,\mathrm{DF}$. And because the ∠ ECB = ACE, ∴ (III. XXI.) ECB = ABE, and CBO = ABO; hence (I. XXXII.) EOB = EBO; ∴ EB = EO; but the ∠ OBO‴ is right, ∴ the ∠ˢ BOO″, BO″O are together equal to a right ∠; but EOB = EBO; ∴ EO‴B = EBO‴; ∴ EB = EO‴; hence OO‴ is bisected in E, and EL is parallel to O‴K; ∴ (I. XL., Ex. 3) OK is bisected in L, and divided unequally in M; hence KM − OM = 2 LM; that is, $r''' - r = 2\,\mathrm{EF}$; and we have proved $r' + r'' = 2\,\mathrm{DF}$; ∴ $r' + r'' + r''' - r = 2\,\mathrm{DE} = 4\,\mathrm{R}$. Hence $r' + r'' + r''' = 4\,\mathrm{R} + r$.

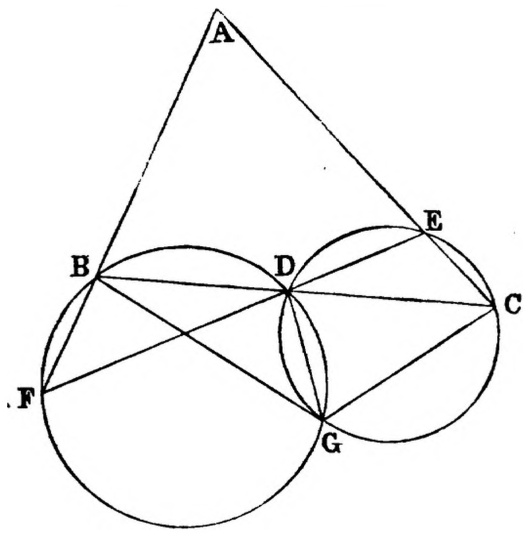
(2) It has been shown that $r''' - r = 2\,\mathrm{EF}$; but EF is $= \mu$; hence $r''' - r = 2\mu$. Similarly $r - r = 2\mu'$, and $r'' - r = 2\mu''$; hence $r' + r'' + r''' - 3r = 2(\mu + \mu' + \mu'')$, that is, $4\,\mathrm{R} + r - 3r = 2(\mu + \mu' + \mu'')$. And therefore $(\mu + \mu' + \mu'') = 2\,\mathrm{R} - r$.

(3) **Dem.**—$\mu + \delta = \mathrm{R}$, $\mu' + \delta' = \mathrm{R}$, and $\mu'' + \delta'' = \mathrm{R}$; hence we



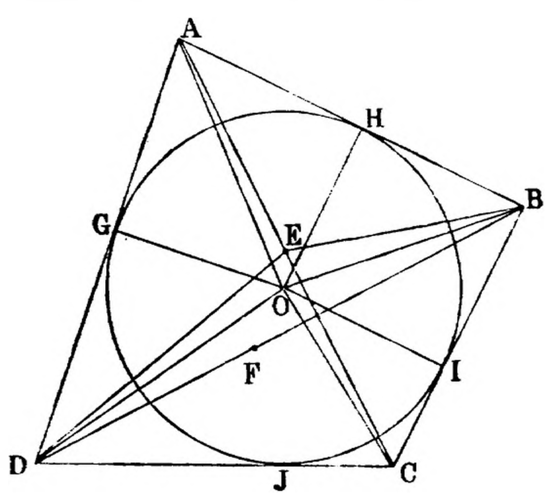
have $\mu + \mu' + \mu'' + \delta + \delta' + \delta'' = 3\,\mathrm{R}$; that is, $2\,\mathrm{R} - r + \delta + \delta' + \delta'' = 3\,\mathrm{R}$. And hence $\delta + \delta' + \delta'' = \mathrm{R} + r$.

20. **Dem.**—Let G be the second point of intersection. Join GB, GC, GD. Now (III. xxii.) the sum of the ∠s DGC, DEC is two right ∠s; but DEC = EAF + AFE, and AFE = BGD



(III. xxi.); ∴ BGC + BAC is equal to two right ∠s; hence BACG is a cyclic quadrilateral; ∴ the circle through B, A, C will pass through G. And the locus of G is a circle.

21. Let ABCD be the quadrilateral, E, F the middle points of the diagonals, and O the centre of the inscribed ⊙. It is required to prove that the points E, O, F are collinear.

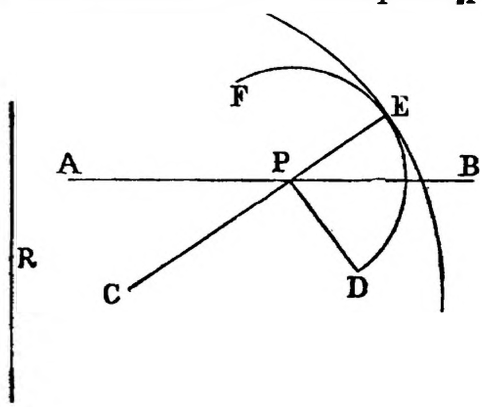


Dem.—Join EB, ED, and join O to the points of contact G, H, I, J.

Now (I. xxxviii.) the $\triangle$ ABE = CBE, and the $\triangle$ ADE = CDE; $\therefore$ AEB + CDE = $\frac{1}{2}$ ABCD; hence the sum of the areas of AEB and CDE is given, and their bases AB, CD are given; $\therefore$ (i., Ex. 29) the locus of E is a straight line, and F is a point on the locus; since it can be shown in the same manner that AFB + CFD = $\frac{1}{2}$ ABCD. Again, the $\triangle$ OAG = OAH, and OIB = OBH; $\therefore$ the area of OAB is half the area of the figure GABIO. Similarly, OCD = $\frac{1}{2}$ GOICD; hence OAB + OCD = $\frac{1}{2}$ ABCD, and $\therefore$ (i., Ex. 29) O is a point on the locus; that is, the points E, O, F are on the same straight line.

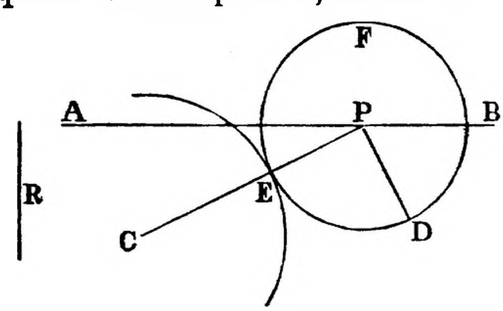
22. (1) Let AB be a given line; C, D two given points. It is required to find a point P on AB, so that CP + DP = R (a given line).

Sol.—With C as centre, and a radius equal to R, describe a $\odot$, and describe a second $\odot$ DEF, having its centre on AB, passing through D, and touching the first $\odot$ internally in E (III. xxxvii., Ex. 3). Let P be its centre. P is the required point.



Dem.—Join CP, and produce it; then (III. xi.) CP produced passes through E. Join PD. Now PE = PD; $\therefore$ CP + PD = CE = R.

(2) It is required to find a point P, so that CP − DP = R.



Sol.—With C as centre, and a radius equal to R, describe a $\odot$,

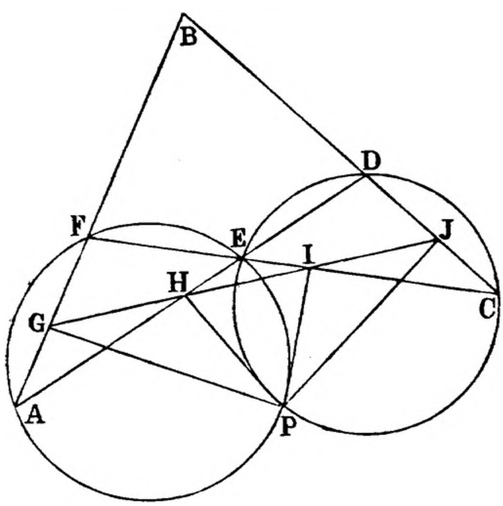
and describe a second ⊙ DEF, having its centre on AB, passing through D, and touching the first ⊙ externally in E. Let P be its centre. P is the required point.

Dem.—Join CP, DP. Now CP = CE + EP; ∴ CP − EP = CE = R; that is, CP − DP = R.

23. Let AB, AD, CB, CF be the four lines. About the △s AFE, CDE describe ⊙s; let them intersect in P. P is the point required.

Dem.—From P let fall ⊥s PG, PH, PI, PJ on the sides of the △s AFE, CDE.

Now (III. xxii., Ex. 12) the feet of the ⊥s on the sides of the △ AFE are collinear. Similarly the feet of the ⊥s on the sides



of the △ CDE are collinear. Hence the feet of the ⊥s PG, PH, PI, PJ are collinear; and these are the ⊥s on the four given lines AB, AD, CB, CF.

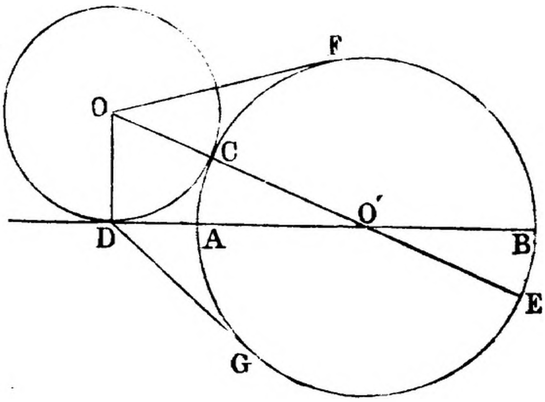
24. See "Sequel to Euclid," Book III., Prop. xiv.

25. See "Sequel to Euclid," Book III., Prop. xiv., Cor.

26. (1) See "Sequel to Euclid," Book III., Prop. v.

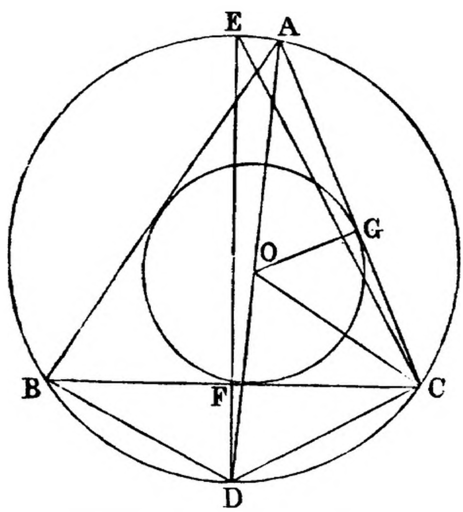
(2) **Dem.**—Let AB be the diameter of the semicircle ACB. Produce BA to D, and let a ⊙ whose centre is O touch ACB in C, and BD in D. From O let fall a ⊥ OD on BD. Join OO′. OO′ passes through C (III. xii.). Produce OO′ to meet ACB in E, and from O, D draw OF, DG tangents to ACB. Now EO . OC = OF^2 (III. xxxvi.) = $OD^2 + DG^2$ ("Sequel," Book III.,

Prop. xxi.), and $OC^2 = OD^2$. Subtracting, we get $(EO - OC)\ OC$; that is, $EC . OC = DG^2$; that is, $2\,Rr = DG^2 = DA . DB$.



27. *Lemma.*—If a △ ABC have a ⊙ inscribed in it, and another circumscribed to it, the rectangle contained by the diameter of the circumscribed ⊙ and the radius of the inscribed ⊙ is equal to the rectangle contained by the segments of any chord of the circumcircle passing through the centre of the inscribed circle.

Dem.—Let O be the centre of the inscribed ⊙. Join AO, and produce it to meet the circumscribed ⊙ in D. From D let fall a ⊥ DF on BC, and produce it to meet the circumference in E. Join EC, OG, OC, BD, CD. Now the arc BD = CD; ∴ the

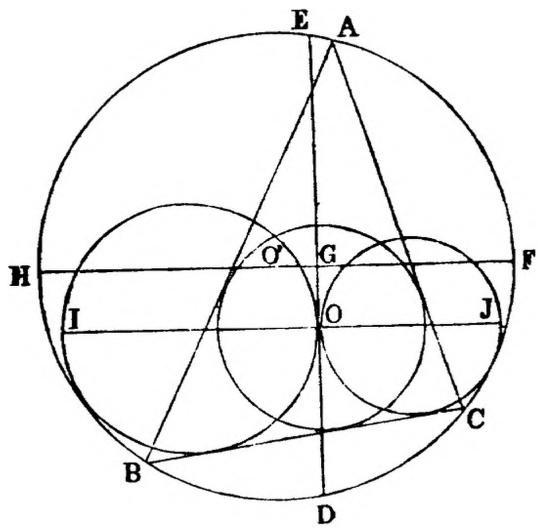


chord BD = CD; hence BF = CF; ∴ DE is the diameter of the circumscribed ⊙; ∴ the ∠ DCE is right, and (III. xviii.) the ∠ OGA is right, and (III. xxi.) the ∠ DEC = OAG;

hence the $\triangle$s DEC, OAG are equiangular; $\therefore$ (III. xxxv., Cor. 3) ED . OG = OA . DC; but DC = DO (Dem., Ex. 19 (1)). Hence ED . OG = OA . OD.

Let ABC be the $\triangle$, O, O′ the centres of the inscribed and circumscribed $\odot$s, and ρ, ρ' the radii of two $\odot$s, touching each other at O, and touching the circumscribed $\odot$. It is required to prove that $\frac{1}{\rho} + \frac{1}{\rho'} = \frac{2}{r}$, r being the radius of the inscribed circle.

Dem.—Through O draw a common tangent to those $\odot$s, meeting the circumscribed $\odot$ in D, E. Join the centres of the $\odot$s whose radii are ρ, ρ', and produce to meet the circumferences



in I, J. Through O′ draw FH ∥ to IJ, and cutting DE in G. Now FG . 2ρ = EO . OD ("Sequel," Book III., Prop. vi.);

$\therefore$ FG = $\frac{\text{EO . OD}}{2\rho}$. Similarly, HG = $\frac{\text{EO . OD}}{2\rho'}$; $\therefore$ FH = $\frac{\text{EO . OD}}{2\rho}$ + $\frac{\text{EO . OD}}{2\rho'}$. Again, 2 R$r$ = EO . OD (*Lemma*); $\therefore$ 2 R = $\frac{\text{EO . OD}}{r}$;

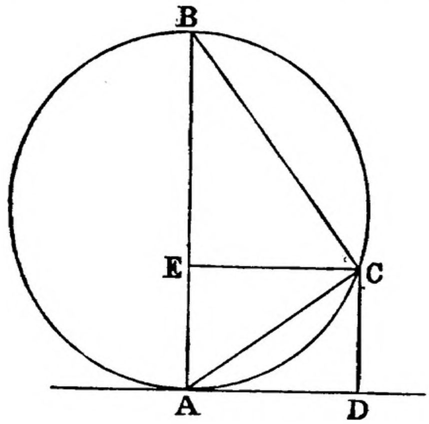
$$\therefore \frac{\text{EO . OD}}{2\rho} + \frac{\text{EO . OD}}{2\rho'} = \frac{\text{EO . OD}}{r}; \text{ therefore } \frac{1}{2\rho} + \frac{1}{2\rho'} = \frac{1}{r};$$

$$\therefore \frac{1}{\rho} + \frac{1}{\rho'} = \frac{2}{r}.$$

28. *Lemma.*—Let AB be the diameter of a $\odot$, AD a tangent.

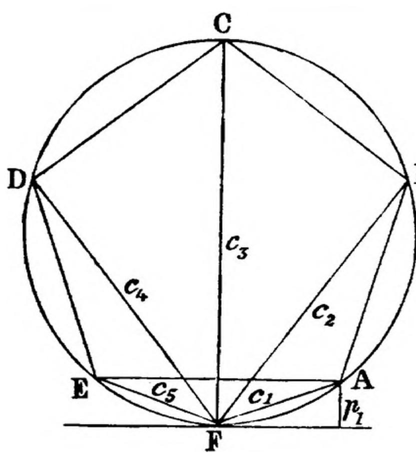
From C, any point in the circumference, a ⊥ CD is let fall on AD, and AC joined. It is required to prove that $AB.CD = AC^2$.

Dem.—Through C draw CE ∥ to AD. Join BC. Now



(I. XLVII., Ex. 1) $AB.AE = AC^2$; but $AE = CD$; ∴ $AB.CD = AC^2$.

Dem.—Let the polygon be a regular pentagon ABCDE. Take any point F in the circumference. At F draw a tangent to the ⊙. Join F to the angular points of the pentagon, and let the joining



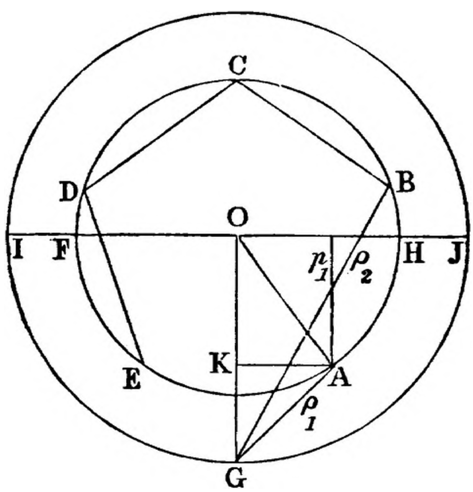
lines be denoted by c_1, c_2, &c. From the angular points let fall ⊥ˢ p_1, p_2, &c., on the tangent, and let the radius be denoted by R.

Now (*Lemma*) $2\,Rp_1 = c_1^2$, and $2\,Rp_2 = c_2^2$, &c.; ∴ $2\,R\,(p_1 + p_2 \ldots\ldots p_5) = (c_1^2 + c_2^2 \ldots\ldots c_5^2)$; but $(p_1 + p_2 \ldots\ldots p_5) = 5\,R$ (Ex. 18); ∴ $10\,R^2 = (c_1^2 + c_2^2 \ldots\ldots c_5^2)$. And similarly for a figure of any number of sides.

29. This is a special case of the next exercise.

30. If any point G in the circumference of any concentric ⊙ be joined to the angular points of an inscribed regular polygon, the sum of the squares of the joining lines is equal to *n* times the square of the radius of the concentric ⊙, together with *n* times the square of the radius of the circumscribed ⊙ ; that is, $\rho_1^2 + \rho_2^2 + \ldots . \rho_5^2 = 5\,R^2 + 5\,r^2$.

Dem.—Let O be the common centre. Through O draw the



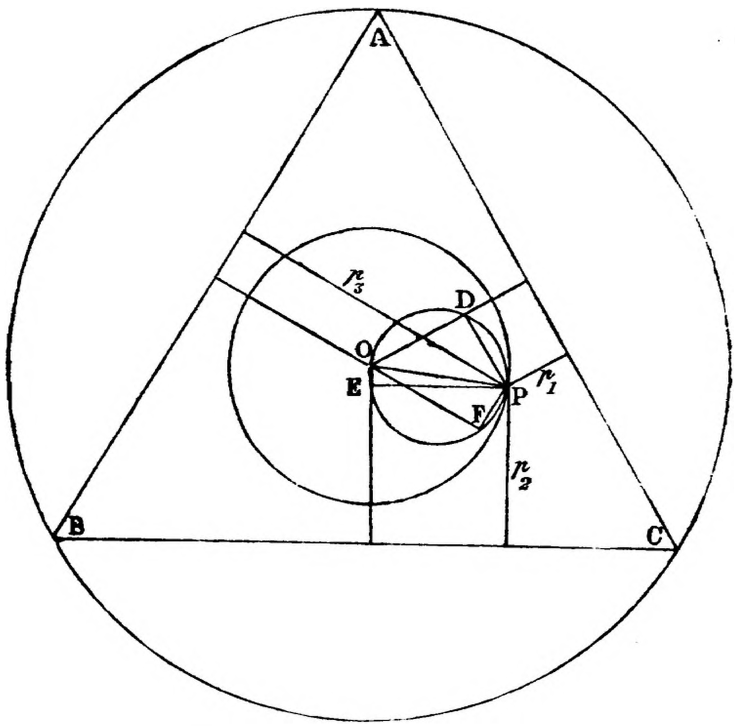
diameter. From A let fall a ⊥ p_1 on IJ, and draw AK parallel to IJ.

Now $AG^2 = OG^2 + OA^2 - 2\,OG\,.\,OK$ (II. XIII.) ; that is, $\rho_1^2 = R^2 + r^2 - 2\,Rp_1$. Similarly, $\rho_2^2 = R^2 + r^2 + 2\,Rp_2$, &c., the sign of $2\,Rp_2$ being positive, since the ⊥ is let fall from above the line. Adding, we get, since the terms by which 2 R is multiplied cancel each other, $\rho_1^2 + \rho_2^2 + \ldots . \rho_5^2 = 5\,(R^2 + r^2)$.

31. Let ABC be an equilateral △ inscribed in a ⊙. From P, any point in the circumference of a concentric ⊙, ⊥s p_1, p_2, p_3, are let fall on the sides of ABC. It is required to prove that $p_1^2 + p_2^2 + p_3^2$ = three times the square of the radius of the inscribed ⊙, together with three half times the square of the radius of the concentric circle.

Dem.—From O, the centre, let fall ⊥s on the sides of ABC. Through P draw PD ∥ to AC, meeting the ⊥ from O on AC in D ; draw PE ∥ to BC, meeting the ⊥ from O on BC in E. Produce the ⊥ from O on AB to F, and draw PF ∥ to AB. Join OP. Now, since the ∠s ODP, OEP, OFP are right, the

⊙ on OP as diameter will pass through D, E, F; and because PD is ∥ to AC, and PE ∥ to BC, ∴ (I. xxix., Ex. 8) the ∠ DPE = ACB = an ∠ of an equilateral △; ∴ DE is $\frac{1}{3}$ of the circumference of DEF. In like manner, EF, DF are each $\frac{1}{3}$ of the circumference of DEF; ∴ D, E, F are the angular points of an

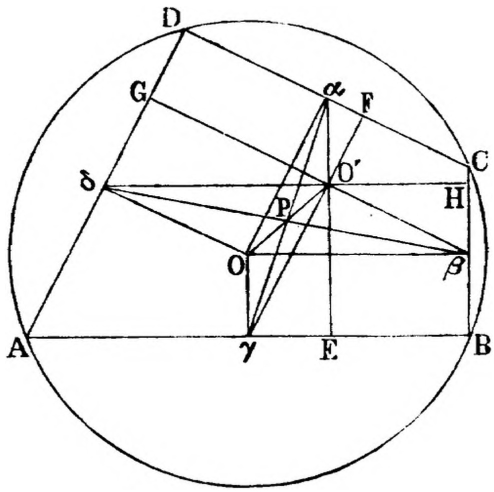


equilateral △ inscribed in DEF, and ∴ (Ex. 28) $OD^2 + OE^2 + OF^2 = 6\left(\frac{OP}{2}\right)^2 = \frac{3\,OP^2}{2}$. Again, $p_1 = (r - OD)$, r being the radius of the inscribed ⊙; ∴ $p_1^2 = r^2 - 2r \cdot OD + OD^2 = (r^2 + OD^2) - 2r\,(r - p_1)$, and $p_2^2 = (r^2 + OE^2) - 2r\,(r - p_2)$, and $p_3^2 = (r^2 + OF^2) - 2r\,(r - p_3)$; ∴ $p_1^2 + p_2^2 + p_3^2 = 3r^2 + \frac{3\,OP^2}{2} - 2r\,\{3r - (p_1 + p_2 + p_3)\}$; but $(p_1 + p_2 + p_3) = 3r$ (Ex. 17). Hence $p_1^2 + p_2^2 + p_3^2 = 3r^2 + \frac{3\,OP^2}{2}$. And in general, in the case of a figure of n sides, the sum of the squares of the ⊥s will equal $nr^2 + \frac{n\,OP^2}{2}$.

32 & 33. These are special cases of Ex. 31.

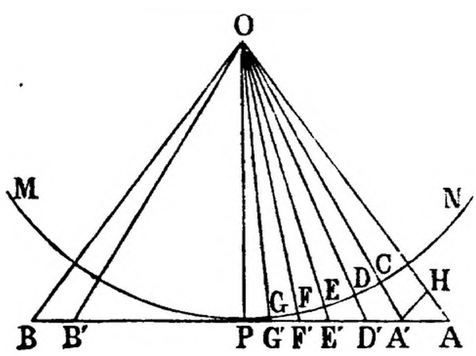
35. Let A, B, C, D be the four concyclic points. From O, the centre of the ⊙, let fall ⊥ˢ $O\alpha$, $O\beta$, $O\gamma$, $O\delta$ on the sides of ABCD; then (III. III.) the sides of the quadrilateral are bisected in α, β, γ, δ. From α, γ let fall ⊥ˢ αE, γF on AB, CD, and let them intersect in O′. Join $\beta O'$, and produce it to meet AD in G. It is required to prove that βG is ⊥ to AD.

Dem.—Join $\alpha\gamma$, $\beta\delta$, OO'. Now, because α, β, γ, δ are the angular points of a parallelogram (I. XL., Ex. 6), and that $\alpha O\gamma O'$



is a parallelogram; ∴ (I. XXXIV., Ex. 1) the lines $\alpha\gamma$, $\beta\delta$, OO' bisect each other. Let P be their common point. Now, in the △ˢ $OP\delta$, $O'P\beta$ we have the sides OP and $P\delta$ equal to O′P and $P\beta$, and the contained ∠ˢ equal; ∴ the ∠ $O\delta P$ = $O'\beta P$; ∴ βG is ∥ to $O\delta$; ∴ βG is ⊥ to AD. Similarly, if we join $\delta O'$ and produce it, δH will be ⊥ to BC.

36. Let MN be an arc of a ⊙ whose centre is O. Let AB be



the side of a regular pentagon, and A′B′ the side of a regular

hexagon circumscribed about it. It is required to prove that the perimeter of the pentagon is greater than that of the hexagon.

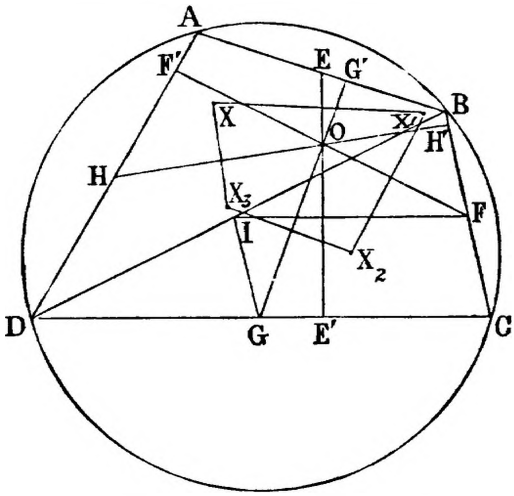
Dem.—Let AB touch MN in P. Join OA, OB, OA′, OB′, OP. Now (XII.) the △ AOP = BOP; ∴ the ∠ AOP = ½ AOB; but AOB = $\frac{4\ rt.\ \angle^s}{5}$; ∴ AOP = $\frac{2\ rt.\ \angle^s}{5}$. In like manner the ∠ A′OP = $\frac{2\ rt.\ \angle^s}{6}$; ∴ the ∠ AOA′ = $\frac{2\ rt.\ \angle^s}{30}$.

Let C be the point where OA′ cuts the ⊙. Then if we divide the arc CP into five equal parts in the points D, E, F, G, join OD, &c., and produce to meet AB in the points D′, E′, F′, G′, the ∠^s A′OD′, D′OE′, &c., will be each $\frac{1}{30}$ of two right ∠^s. Again, the line OA is greater than OD′ (I. XIX., Ex. 4). Cut off OH = OD′. Join A′H. Then (I. IV.) A′D′ = A′H, and the ∠ OD′A′ = OHA′; ∴ the ∠ OD′E′ = AHA′; but OD′E′ is greater than OAD′ (I. XVI.); ∴ AHA′ is greater than A′AH; and hence AA′ is greater than A′H; that is, than A′D′. Similarly, A′D′ is greater than D′E′; D′E′ greater than E′F′, &c.; hence 5 AA′ is greater than A′P. To each add 5 A′P, and we have 5 AP greater than 6 A′P; ∴ 5 AB is greater than 6 A′β′; but 5 AB is the perimeter of the pentagon, and 6 A′B′ that of the hexagon. Hence the perimeter of the pentagon is greater than that of the hexagon; and in general the greater the number of sides, the less the perimeter.

37. By the last exercise the area of a pentagon is less than the area of a square; but the area of a square is equal to the square of the diameter. Hence the area of a pentagon is less than the square of the diameter.

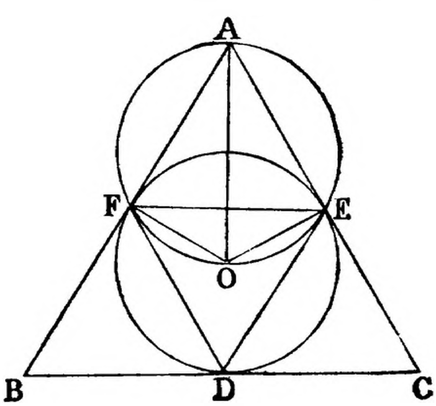
38. **Dem.**—Join the four concyclic points A, B, C, D. Bisect the joining lines in E, F, G, H. Join BD, and bisect it in I. Then (V., Ex. 4) the nine-points ⊙ of the △ ABD will pass through the points H, E, I. Similarly, the nine-points ⊙ of the △ ABC will pass through E, F, and the middle point of AC. Hence two of the nine-points ⊙^s will pass through E. In like manner two of them will pass through each of the points F, G, H. From E, F, G, H let fall ⊥^s EE′, FF′, GG′, HH′ on the opposite sides; these ⊥^s will co-intersect in a point O (Ex. 35). Join IF, IG. Now, because each of the ∠^s AG′O, AF′O is right, ∴ the ∠^s F′AG′, F′OG′ are together equal to two right ∠^s, and the ∠^s BAD, BCD are equal to two right ∠^s; ∴ the ∠ F′OG′ = BCD; that is, the ∠ FOG = BCG; but (I. XXXIV.) BCG

= FIG; ∴ FOG = FIG; and hence the ⊙ through the points F, G, I, must pass through O. In like manner each of the four nine-points ⊙s must pass through O. Now, since two of these ⊙s pass through E and O, if we bisect EO, and erect XX_1 ⊥ to it, their centres must be in XX_1. Similarly, the centres of each other pair must be in the lines X_1X_2, X_2X_3, X_3X_1. Hence the points X, X_1, X_2, X_3 must be the centres. And because each of the lines



XX_1, CD is ⊥ to EE′, they are parallel to each other. Similarly, the remaining sides of $XX_1X_2X_3$ are parallel to the remaining sides of ABCD; hence the ∠s X and X_2 are equal to the ∠s A and C; but A and C are together equal to two right ∠s, ∴ X and X_2 are equal to two right ∠s. Hence the points X, X_1, X_2, X_3 are concyclic.

39. Let AB, AC be two fixed lines, having their included



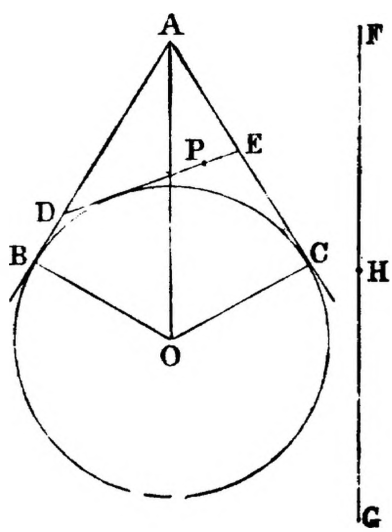
∠ BAC equal to an ∠ of an equilateral △; and let BC be a

third line forming a △ with AB, AC. Bisect BC, AC, AB in D, E, F. Join DE, EF, DF. The ⊙ through D, E, F is the nine-points ⊙ of the △ ABC (v., Ex. 4). It is required to prove that the locus of its centre O is a right line.

Dem.—Join OA, OE, OF. Now DE, DF are respectively ∥ to AB, AC (I. xl., Ex. 2); ∴ AEDF is a parallelogram; ∴ the ∠ FDE = FAE; but FOE = 2 FDE (III. xx.); ∴ FOE = 2 FAE; hence FOE is twice an ∠ of an equilateral △; ∴ FOE + FAE are equal to two right ∠s; hence FAOE is a cyclic quadrilateral. Again, because OE = OF, the arc OE = OF, and ∴ (III. xxvi.) the ∠ OAE = OAF; ∴ the ∠ FAE is bisected. Hence the line OA is given in position; and since O is a point on it, the locus of O is a right line.

41. Let AB, AC be two lines given in position, P a given point, and let the line FG be equal to the given perimeter. It is required to draw a transversal through P, so that the △ DAE shall have a perimeter equal to FG.

Sol.—Bisect FG in H. In AB take AB = GH, and erect BO ⊥ to AB. Bisect the ∠ BAC by AO, and let fall a ⊥ OC on AC. Then (I. xxvi.) the △s ABO, ACO are equal in every



respect; ∴ OB = OC; hence the ⊙, with O as centre and OB as radius, will pass through C, and touch the lines AB, AC in B, C. Through P draw DE, touching this ⊙, and cutting AB, AC in D, E. ADE is the △ required. For (iv., Ex. 4) AB is

equal to half the perimeter of ADE. Hence the perimeter is equal to 2 AB, or FG.

42. (1) Let BAC be the vertical ∠, X its bisector, and FG the perimeter.

Sol.—Bisect the ∠ BAC by AP, and make AP = X. Through P draw DE, cutting off a △ ADE whose perimeter is equal to FG (Ex. 41).

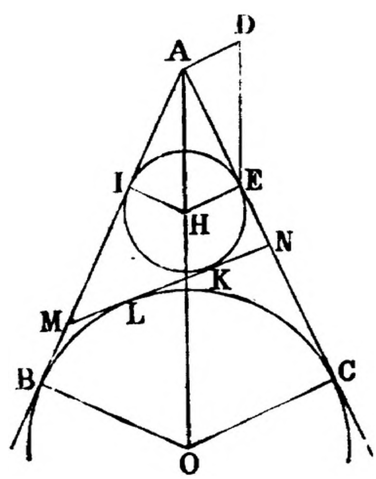
(2) Let BAC be the vertical ∠, FG the perimeter, and X the perpendicular.

Sol.—Bisect FG in H, take AB = GH, erect BO ⊥ to AB, bisect the ∠ BAC by AO, and from O let fall OC ⊥ on AC; then the ⊙, with O as centre, and OB as radius, will pass through C, and will touch AB, AC, in B, C. With A as centre, and a radius equal to X, describe a ⊙, cutting AB, AC in M, N. Draw a common tangent to the two ⊙s, meeting AB, AC in D, E. ADE is the required triangle.

Dem.—Join AP, P being the point where DE touches the ⊙ MN. Now (III. xviii.) the ∠ APE is right, ∴ AP is a ⊥, and it is equal to X; and, as in Ex. 41, the perimeter of the △ ADE = FG.

(3) Let BAC be the vertical ∠, FG the perimeter, and R the radius of the inscribed circle.

Sol.—Bisect BAC by AO. Draw AD ⊥ to AC, and make it



equal to R. Through D draw DE ∥ to AO, and where it meets AC draw EH ∥ to AD. From H let fall HI ⊥ on AB. Take AB

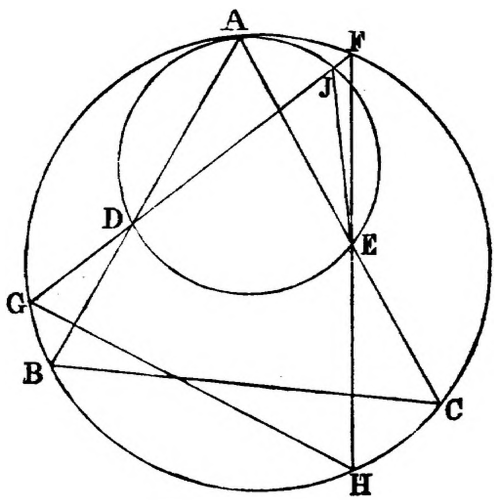
$= \frac{1}{2}$ FG; erect BO $\perp$ to AB, and from O let fall a $\perp$ OC on AC. Now, as in Ex. 41, HE = HI, and OB = OC; hence the ⊙s with H, O as centres, and HE, OC as radii, will pass through the points I, B. Draw a common tangent, touching the ⊙s in K and L, and cutting AB, AC in M, N. AMN is the required triangle.

For, as before, the perimeter of AMN = FG. And since ADEH is a parallelogram, EH = AD = R.

43. (1) Let ABC be the given ⊙, D, E the points. It is required to inscribe a $\triangle$ in ABC, so that two sides may pass through D, E, and the third be a maximum.

Sol.—Describe a ⊙ passing through D, E, and touching ABC in A (III. xxxvii., Ex. 1). Join AD, AE, and produce to meet ABC in B, C. Join BC. ABC is the required triangle.

Dem.—Take any other point F in ABC. Join FD, FE, and produce to meet ABC in GH. Join GH, JE, J being the point



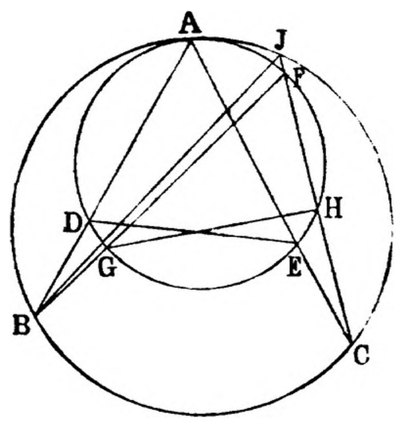
where FG cuts the ⊙ ADE. Now the $\angle$ DJE is greater than DFE; $\therefore$ the $\angle$ DAE is greater than DFE; $\therefore$ the arc BC is greater than GH. Hence the chord BC is greater than GH.

(2) Let ADE be the given ⊙; B, C the points.

Sol.—Through B, C describe a ⊙ ABC, touching ADE in A. Join AB, AC, cutting the ⊙ ADE in D, E. Join DE. ADE is the required triangle.

Dem.—Take any point F in ADE. Join BF, CF, cutting the ⊙ ADE in GH. Join GH. Produce CF to meet ABC in J. Join BJ. Now the $\angle$ BFC is greater than BJC, that is, greater

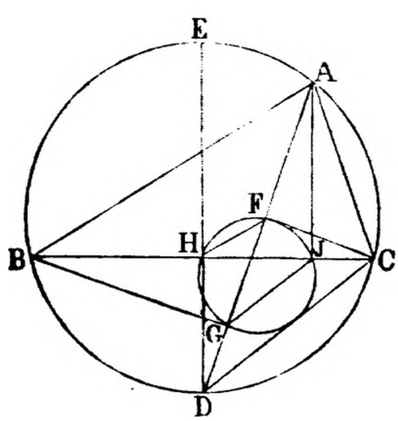
than BAC ; ∴ the arc GH is greater than DE. Hence the chord GH is greater than DE.



44. Let Δ represent the area of the triangle.

Now $r' = \frac{\Delta}{s-a}$ (IV., Ex. 10), $r'' = \frac{\Delta}{s-b}$; ∴ $r'r'' = \frac{\Delta^2}{(s-a)(s-b)}$; but $\Delta^2 = s \,.\, s-a \,.\, s-b \,.\, s-c$ (IV., Ex. 12); therefore $r'r'' = \frac{s \,.\, s-a \,.\, s-b \,.\, s-c}{(s-a)(s-b)} = s \,.\, s-c$. Similarly, $r''r''' = s \,.\, s-a$, and $r'''r' = s \,.\, s-b$. Hence $r'r'' + r''r''' + r'''r' = s\{3s - (a+b+c)\}$; but $(a+b+c) = 2s$ (IV., Ex. 2); ∴ $r'r'' + r''r''' + r'''r' = s\{3s - 2s\} = s \,.\, s = s^2$.

45. Let ABC be a Δ inscribed in a ⊙. Draw the diameter



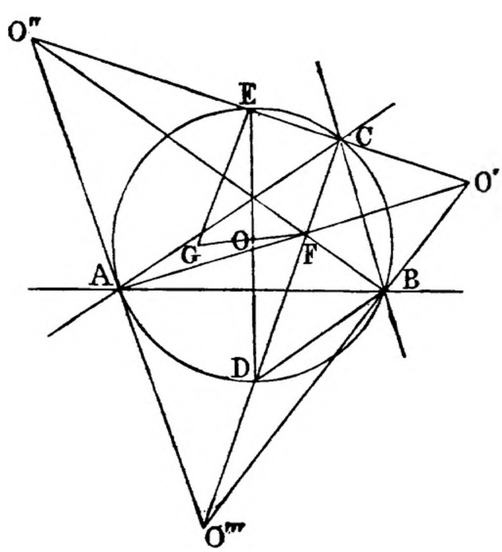
DE. Join AD. AD is the internal bisector of the vertical ∠.

From A let fall a ⊥ AJ on BC. From B and C let fall ⊥s BG, CF on AD, and let H be the point where DE bisects BC. It is required to prove that the points F, H, G, J are concyclić.

Dem.—Join FH, GJ, CD. Now, since each of the ∠s BGA, BJA is right, BGJA is a cyclic quadrilateral; ∴ the ∠ BAG = BJG. And because DHFC is a cyclic quadrilateral, the ∠ DCH = DFH; but (III. xxi.) DCH = BAD; ∴ DFH = BJG. Hence the points F, H, G, J are concyclic.

46. Let ABC be the △ whose base AB and vertical ∠ ACB are given.

Describe a ⊙ about ACB. Let O be its centre. Draw DE, the diameter, perpendicular to AB. Join CD, CE. CD, CE are the internal and external bisectors of the ∠ ACB (III. xxx., Ex. 2). Bisect the external ∠ CAB by AO″, meeting CE produced. Produce CD, O″A to meet in O‴. Join O‴B. Produce

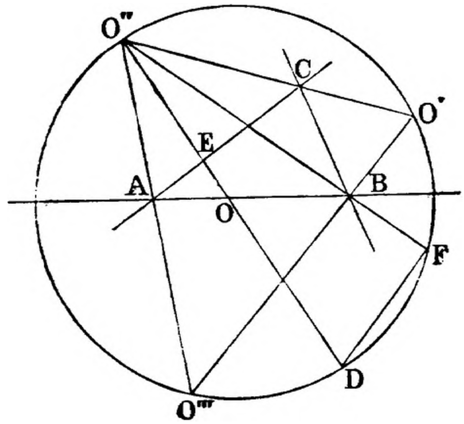


O‴B, O″C to meet in O′. O′B is the external bisector of the ∠ CBA (I. xxvi., Ex. 8); O′, O″, O‴ are the centres of the escribed ⊙s. Join O′A, O″B, intersecting CD in F. Join FO. Draw EG ∥ to CD, meeting FO produced in G. G is the centre of the ⊙ passing through O′, O″, O‴. It is required to find its locus.

Dem.—Join BD. Now, because F is the orthocentre of the $\triangle$ O′O″O‴ (IV. IV., Ex. 6), O the centre of its nine-points $\odot$ (IV. V., Ex. 5), and EG the $\perp$ from the middle point of O′O″, $\therefore$ OF = OG (IV. V., Ex. 4); and since the $\angle$ GEO = FDO (I. XXIX.), and GOE = FOD; $\therefore$ EG = DF; but DF = DB (Dem. of Ex. 27), and DB is given, $\therefore$ EG is given, and the point E is given. Hence the locus of G is a $\odot$, having E as centre and EG as radius.

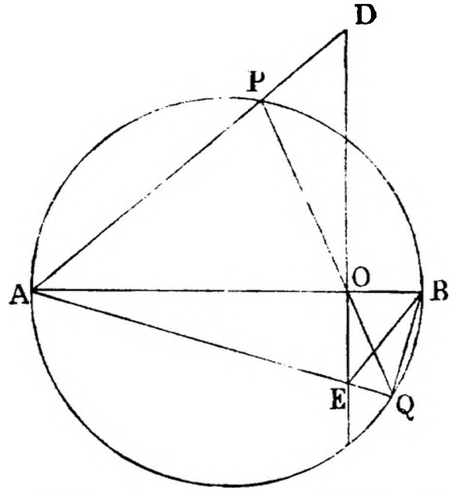
47. Let ABC be the $\triangle$; O′, O″, O‴ the centres of the escribed circles.

Dem.—Describe a $\odot$ about the $\triangle$ O′O″O‴. Let O be its centre. Join O″O, and produce it to meet the circumference in



D, and cutting AC in E. We shall prove that O″O is $\perp$ to AC. Join O″B, and produce it to meet the circumference in F. Join DF. Now the $\angle$ O″FD is right (III. XXXI.), and O″BO‴ is right, since O″B is $\perp$ to O′O‴; $\therefore$ O′O‴ and FD are parallel; $\therefore$ (III. XXVI., Cor. 2) the arc O‴D = O′F; hence the $\angle$ O‴O″D = O′O″F, and the $\angle$ O″AE = O″O′B (I., Ex. 36); $\therefore$ the $\angle$ O″EA = O″BO′; but O″BO′ is right, $\therefore$ O″EA is right; hence O″O is $\perp$ to AC. Similarly, if we join O′O, O‴O, they will be $\perp$ to BC, AB. Hence the three $\perp^s$ are concurrent.

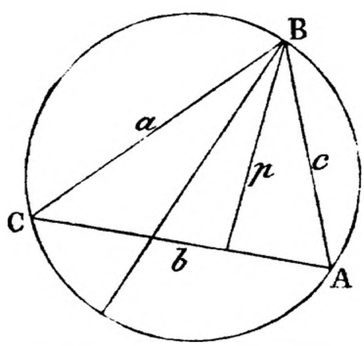
48. **Dem.**—Join BE, BQ, BD. Now (III. XXXI.) the $\angle$ AQB is right, and EOB is right (hyp.); $\therefore$ OEQB is a cyclic quadrilateral; $\therefore$ the $\angle$ OQB = OEB; but OQB = PAB (III. XXI.);



therefore OEB is equal to PAB. And hence the ⊙ through the points A, E, B will pass through D.

49. Let ABC be a △ whose sides a, b, c are in arithmetical progression; a being the greatest, and c the least. It is required to prove that $6\,\mathrm{R}r = ac$.

Dem.—From B let fall a ⊥ p on b, and draw the diameter. Now $2\,\mathrm{R}p = ac$ (III. xxxv., Ex. 1), $\therefore 2\,\mathrm{R}.bp = abc$; but bp is

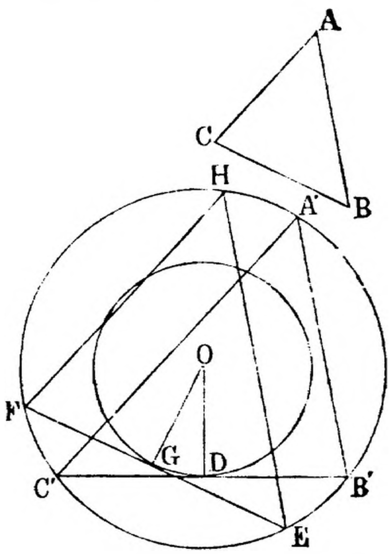


equal to twice the area; that is, equal to 2Δ (suppose), $\therefore 2\mathrm{R}.2\Delta = abc$, $\therefore \mathrm{R} = \dfrac{abc}{4\Delta}$; and since the sides are in A.P., $a + c = 2b$; $\therefore a + b + c = 3b$; but $(a + b + c) = 2s$, therefore $2s = 3b$. Again (IV. Ex. 9), $r = \dfrac{\Delta}{s}$; and multiplying this and the equation $\mathrm{R} = \dfrac{abc}{4\Delta}$ we get $\mathrm{R}r = \dfrac{abc}{4s}$, $\therefore \mathrm{R}r = \dfrac{abc}{6b}$; and hence $6\,\mathrm{R}r = ac$.

50. Let A′B′C′ be the ⊙; and AB, BC, CA three lines in the form of a △. It is required to inscribe in A′B′C′ a △ whose sides shall be parallel to the sides of ABC.

Sol.—Take a point A′ in the circumference, and draw A′B′ ∥ to

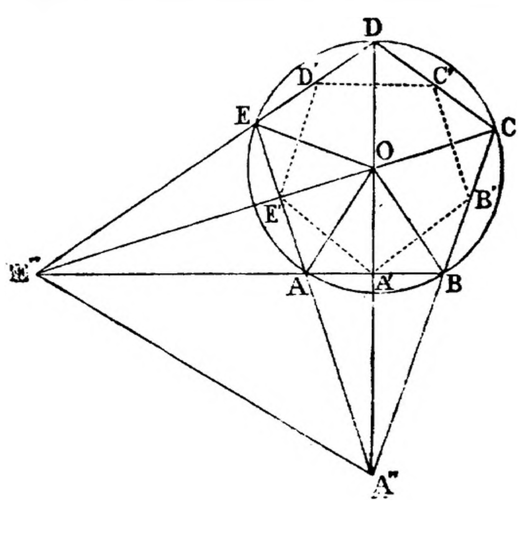
AB, and A′C′ ∥ to AC. Join B′C′. If B′C′ is ∥ to BC, the thing required is done. If not, from the centre O let fall a ⊥ OD on



B′C′. With O as centre, and OD as radius, describe a ⊙. Draw EF, touching this ⊙, and ∥ to BC (III. xvi., Ex. 2). Join O to G, the point of contact. Draw FH ∥ to C′A′, and join HE. HFE is the △ required.

Dem.—Because OG = OD, EF = B′C′ (III. xiv.), ∴ the arc EF = B′C′; hence the arc FC′ = B′E; but FC = HA′ (III. xxvi., Cor. 2), ∴ B′E = HA′, ∴ HE is parallel to A′B′; that is parallel to AB; and FH is parallel to A′C′, that is to AC; and EF is parallel to BC. Hence the sides of the △ HFE are parallel to the sides of ACB.

51. **Dem.**—Let O be the centre of the circumscribed ⊙. Join



DA″, CE″, OA, OB, OE, &c. Now the △ A″OE″ + DOC − (A″OC + E″OD) = 4 A′OE′ (Book I., Ex. 52); that is, AA″E″ + AOA″ + ACE″ + DOC − (BOC + A″OB + EOD + EOE″) = 4 A′OE′; but evidently AOA″ = A″OB, AOE″ = EOE″, and DOC = EOD; ∴ AA″E″ − BOC = 4 A′OE′, and BOC = A′OE′ + A′AE′. Adding, we get AA″E′ − A′AE′ + 5 A′OE′ = pentagon A′B′C′D′E′.

52. (1) **Dem.**—Let ABCDE be the equilateral inscribed polygon.

Now, since the sides are equal, the arcs are equal; therefore the whole arc EABC = DEAB; hence the ∠ CDE = BCD. Similarly, the ∠ BCD = ABC, &c. Hence the polygon is regular.

(2) **Dem.**—Let ABCDE be the equilateral circumscribed polygon; F, G, H, I, J the points of contact, and O the centre. Join OA, OB, OF, OG, OH.

Now ID = HD, ∴ IE = HC, ∴ JE = GC, ∴ AJ = BG, ∴ AF = BF. Now since AF = BF, OF common, and the ∠ AFO = BFO, ∴ the ∠ OAF = OBF; ∴ the ∠ BAE = ABC. Similarly all the ∠ˢ are equal. Hence the polygon is regular.

53. (1) Let ABCDE be the equiangular circumscribed polygon; F, G, H, I, J the points of contact, and O the centre. Join OA, OB, OG, OH.

Now since the ∠ CBA = EAB, their halves are equal; that is, the ∠ OBF = OAF, and the ∠ OFB = OFA, each being right, and the side OF common, ∴ (I. XXVI.) BF = AF; that is, AB = 2 AF. Similarly, AE = 2 AJ; but AF = AJ, ∴ AB = AE. In like manner all the sides are equal. Hence the polygon is regular.

(2) **Dem.**—Let ABCDE be the inscribed polygon, and O the centre. Join OA, OB, OC, OD, OE. Now the ∠ ABC = EAB (hyp.); but the ∠ OBA = OAB, since OA = OB, therefore the ∠ OBC = OAE; that is, OCB = OEA; but the ∠ BCD = AED, ∴ OCD = OED; that is, ODC = ODE. Now, in the △ˢ ODC, ODE, the ∠ˢ OCD, ODC are equal to the ∠ˢ OED, ODE, and the side OD common; hence (I. XXVI.) DC = DE. Similarly all the sides are equal. Hence the polygon is regular.

54. The sum of the $\perp^s$ drawn to the sides of an equiangular polygon X from any point P inside the figure is constant.

Dem.—Suppose a regular polygon Y of the same number as X constructed so as to include X, and have its sides parallel to those of X. Then, if the $\perp^s$ from P on the sides of X be produced to meet the sides of Y, their sum is constant (Book IV., Ex. 17); but the excess of the latter sum over the former is constant. Hence the former sum is constant.

55. Express the sides of a $\triangle$ in terms of the radii of its escribed circles.

If the radii be r', r'', r''', we have, denoting the area of the triangle by Δ (Book IV., Prop. IV., Ex. 10),

$$r' = \frac{\Delta}{s-a},\quad r'' = \frac{\Delta}{s-b},\quad r''' = \frac{\Delta}{s-c};$$

$$\therefore\ r'(r''+r''') = \frac{\Delta^2}{(s-a)(s-b)} + \frac{\Delta^2}{(s-a)(s-c)};$$

but (Book IV., Prop. IV., Ex. 12) $\Delta^2 = s.s-a.s-b.s-c$;

$$\therefore\ r'(r''+r''') = s.s-c + s.s-a = sa,$$

and (Book IV., Ex.) $\quad \sqrt{r'r'' + r''r''' + r'''r'} = s$;

$$\therefore\ a = \frac{r'(r''+r''')}{\sqrt{r'r'' + r''r''' + r'''r'}}.$$

Similarly,

$$b = \frac{r''(r'''+r')}{\sqrt{r'r'' + r''r''' + r'''r'}},$$

$$c = \frac{r'''(r'+r'')}{\sqrt{r'r'' + r''r''' + r'''r'}}.$$

BOOK V.

Miscellaneous Exercises.

1. (1) Let a be greater than b. It is required to prove that $\dfrac{a-x}{b-x}$ is greater than $\dfrac{a}{b}$.

Dem.—Subtract, and we get $\dfrac{ab-bx-ab+ax}{b(b-x)}$; that is $\dfrac{(a-b)x}{b(b-x)}$; but since a is greater than b, $\dfrac{(a-b)x}{b(b-x)}$ is positive. Hence $\dfrac{a-x}{b-x}$ is greater than $\dfrac{a}{b}$.

(2) To prove that $\dfrac{a}{b}$ is greater than $\dfrac{a+x}{b+x}$

Dem.—Subtract, and we get $\dfrac{a}{b}-\dfrac{a+x}{+x}=\dfrac{ab+ax-ab-bx}{b(b+x)}$ $=\dfrac{(a-b)x}{b(b+x)}$; but because a is greater than b, $\dfrac{(a-b)x}{b(b+x)}$ is positive. Hence $\dfrac{a}{b}$ is greater than $\dfrac{a+x}{b+x}$.

2. The proof of this is similar to that of Ex. 1.

3. Let a, b, c, d be the four magnitudes; then if $a:b::c:d$, it is required to prove that $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$.

Dem.—Because $a:b::c:d$, we have $a+b:b::c+d:d$ (XVIII.); that is $\dfrac{a+b}{b}=\dfrac{c+d}{d}$ Again, $a-b:b::c-d:d$ (XVII.); that is, $\dfrac{a-b}{b}=\dfrac{c-d}{d}$. Dividing, we get $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$.

4. Let a, b, c, d, and e, f, g, h, be the two sets of four magnitudes that are proportionals; that is, $a:b::c:d$; and $e:f::g:h$. It is required to prove that $ae:bf::cg:dh$.

Dem.—Because $a : b :: c : d$, we have $\frac{a}{b} = \frac{c}{d}$. Similarly, $\frac{e}{f} = \frac{g}{h}$. Multiplying together, we get $\frac{ae}{bf} = \frac{cg}{dh}$; that is, $ae : bf :: cg : dh$.

5. It is required to prove that $\frac{a}{e} : \frac{b}{f} :: \frac{c}{g} : \frac{d}{h}$.

Dem.—As in (4), we have $\frac{a}{b} = \frac{c}{d}$, and $\frac{e}{f} = \frac{g}{h}$; $\therefore \frac{a}{b} \div \frac{e}{f} = \frac{c}{d} \div \frac{g}{h}$; but $\frac{a}{b} \div \frac{e}{f} = \frac{af}{be} = \frac{a}{e} \div \frac{b}{f}$ and $\frac{c}{d} \div \frac{g}{h} = \frac{ch}{dg} = \frac{c}{g} \div \frac{d}{h}$; $\therefore \frac{a}{e} \div \frac{b}{f} = \frac{c}{g} \div \frac{d}{h}$. Hence $\frac{a}{e} : \frac{b}{f} :: \frac{c}{g} : \frac{d}{h}$.

6. Let a, b, c, d be the four magnitudes.

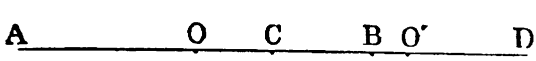
Dem.—$a : b :: c : d$, $\therefore \frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a^2}{b^2} = \frac{c^2}{d^2}$; that is, $a^2 : b^2 :: c^2 : d^2$. Similarly $a^3 : b^3 :: c^3 : d^3$.

7. Let a, b, c, d; a, b, c, d', be the two sets of magnitudes. It is required to prove that $d = d'$.

Dem.—$a : b :: c : d$, and $a : b :: c : d'$, $\therefore \frac{a}{b} = \frac{c}{d}$, and $\frac{a}{b} = \frac{c}{d'}$; $\therefore \frac{c}{d} = \frac{c}{d'}$. Hence $d = d'$.

8. **Dem.**—Since the three magnitudes are continual proportions, we have $\frac{a}{b} = \frac{b}{c}$, and $\frac{b}{c} = \frac{b}{c}$. Multiplying these equalities, we get $\frac{a}{c} = \frac{b^2}{c^2}$; that is, $a : c :: b^2 : c^2$. Again, $\frac{a}{b} = \frac{b}{c}$; $\therefore \left(\frac{a}{b} - 1\right) = \left(\frac{b}{c} - 1\right)$; $\therefore \frac{a-b}{b} = \frac{b-c}{c}$, and therefore $\frac{(a-b)^2}{b^2} = \frac{(b-c)^2}{c^2}$: that is, $(a-b)^2 : (b-c)^2 :: b^2 : c^2$. Hence we have $a : c :: (a-b)^2 : (b-c)^2$.

9. **Dem.**—AC : CB :: AD : DB (hyp.), $\therefore$ AC − CB : AC



+ CB :: AD − DB : AD + DB; that is, 2OC : 2OB :: 2OB : 2OD. Hence OC : OB :: OB : OD.

10. **Dem.**—Because CD is bisected in O′, and produced to O, we have (II. VI.) $OD.OC + O'C^2 = OO'^2$; but $OD.OC = OB^2$ (Ex. 9); $\therefore OB^2 + O'C^2 = OO'^2$; that is, $OO'^2 = OB^2 + O'D^2$.

11. **Dem.**—$AC : CB :: AD : DB$ (hyp.), $\therefore AC : AB - AC :: AD : AD - AB$, $\therefore \dfrac{AC}{AB - AC} = \dfrac{AD}{AD - AB}$, $\therefore AC\,(AD - AB) = AD\,(AB - AC)$; $\therefore AC.AD - AC.AB = AD.AB - AD.AC$.

Transposing, we get $2\,AC.AD = AB\,(AC + AD)$. Divide by $AB.AC.AD$, and we have $\dfrac{2}{AB} = \dfrac{1}{AD} + \dfrac{1}{AC}$.

12. **Dem.**—$BD : BC :: AD : AC$ (hyp.). Working, as in Ex. 11, we get $\dfrac{2}{CD} = \dfrac{1}{BD} + \dfrac{1}{AD}$.

13. **Dem.**—$AC : CB :: AD : BD$ (hyp.), $\therefore AC.BD = CB.AD$, $\therefore AC.BD + CB.AD = 2\,CB.AD$. Again, $AB.CD = (AC + CB)\,(CB + BD) = AC.BD + AC.CB + CB^2 + CB.BD = AC.BD + CB\,(AC + CB + BD) = AC.BD + CB.AD = 2\,CB.AD$.

BOOK VI.

PROPOSITION II.

1. Let AE, BF be two lines cut by the parallels AB, CD, EF. It is required to prove that AC : CE : : BD : DF.

Dem.—Join BE, cutting CD in G. Now in the $\triangle$ AEB we have AC : CE : : BG : GE (II.); and in the $\triangle$ BEF, BG : GE : : BD : DF. Hence AC : CE : : BD : DF.

PROPOSITION III.

2. (1) **Dem.**—Through C draw CF $\parallel$ to AD. Now (I. XXIX.) the $\angle$ EAD = EFC, and the $\angle$ DAC = ACF; but EAD = DAC (hyp.); $\therefore$ AFC = ACF; $\therefore$ AC = AF. Now (II.) BA : AF : : BD : DC; that is, BA : AC : : BD : DC.

(2) **Dem.**—Produce AB through A, cut off AE = AC, and join DE. Now (I. IV.) the $\triangle$s EAD, CAD are congruent; $\therefore$ DE = DC, and the $\angle$ ADE = ADC; hence BD : DE : : BA : AE; that is, BD : DC : : BA : AC.

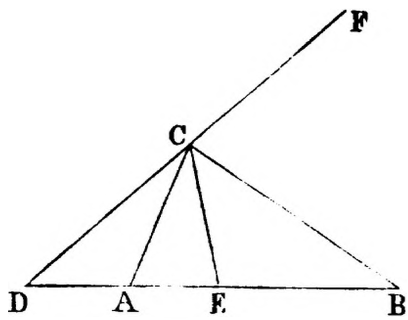
3. Let AB be the base, ACB the vertical $\angle$, CO, CO′ the internal and external bisectors. It is required to prove that AB is divided harmonically in O and O′.

Dem.—In the $\triangle$ ACB, AO : OB : : AC : CB (III.); and in the same $\triangle$, AO′ : O′B : : AC : CB (Ex. 1); $\therefore$ AO : OB : : AO′ : O′B. Hence AB is divided harmonically in O and O′.

4. This is the same as Ex. 3.

5. Let ACB be the right $\angle$, AB the line intersecting the sides CA, CB, and CD, CE any two lines making equal $\angle$s with CA. Produce BA to meet CD. It is required to prove that AB is cut harmonically in E and D.

Dem.—Produce DC to F. Now in the $\triangle$ DCE, DA : AE : : DC : CE (III.). Again, since the $\angle$ ACB is right, the $\angle$s DCA, BCF are together equal to a right $\angle$; but DCA = ACE; $\therefore$ ECB = BCF; $\therefore$ BC is the bisector of the external $\angle$ ECF;

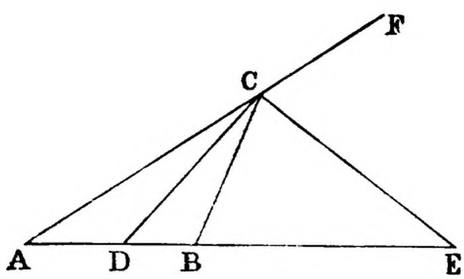


hence (Ex. 1) DC : CE : : DB : BE; ∴ DA : AE : : DB : BE. Hence AB is cut harmonically in E and D.

6. Let AB be the base, AC and CB the sides.

Sol.—Bisect the ∠ ACB by CD. Produce AC to F, and bisect the ∠ BCF by CE, meeting AB produced in E.

Now AD : DB : : AC : CB (III.); but the ratio AC : CB is given (hyp.); ∴ the ratio AD : DB is given; ∴ D is a given point. Again, AC : CB : : AE : EB (Ex. 1); ∴ the ratio AE : EB is given, and AB is given; hence the point E is given. And



because the ∠ ACD = BCD, and FCE = BCE, the ∠ DCE is right; hence the ⊙ on DE as diameter will pass through C; and because the points D, E are given, it will be a given ⊙. It divides the base in the points D, E harmonically, in the ratio of AC : CB, and is the locus of the vertex. It is called the "Apollonian locus."

7. **Dem.**—$b : c :: \text{CD} : \text{DB}$ (III.); $\therefore b + c : c :: \text{CD} + \text{DB} : \text{DB}$; but $\text{CD} + \text{DB} = \text{CB} = a$; $\therefore b + c : c :: a : \text{DB}$; $\therefore (b + c)\,\text{DB} = ac$; hence $\text{DB} = \frac{ac}{b + c}$. Similarly, $\text{D}'\text{B} = \frac{ac}{b - c}$.

Adding, we get $\text{DD}' = \frac{ac}{b + c} + \frac{ac}{b - c} = \frac{2\,abc}{b^2 - c^2}$.

8. (1) **Dem.**—In the last Exercise we got $DD' = \frac{2abc}{b^2 - c^2}$.

$\therefore \frac{1}{DD'} = \frac{b^2 - c^2}{2abc}$. Similarly, $\frac{1}{EE'} = \frac{c^2 - a^2}{2abc}$, and $\frac{1}{FF'} = \frac{a^2 - b^2}{2abc}$.

Adding, we get $\frac{1}{DD'} + \frac{1}{EE'} + \frac{1}{FF'} = 0$.

(2) **Dem.**—From (1) we have

$$\frac{1}{DD'} = \frac{b^2 - c^2}{2abc}, \therefore \frac{a^2}{DD'} = \frac{a^2b^2 - c^2a^2}{2abc}.$$

Similarly,

$$\frac{b^2}{EE'} = \frac{b^2c^2 - a^2b^2}{2abc}, \text{ and } \frac{c^2}{FF'} = \frac{c^2a^2 - b^2c^2}{2abc}.$$

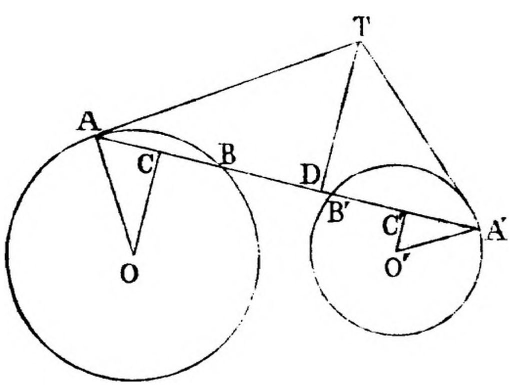
Adding, we have

$$\frac{a^2}{DD'} + \frac{b^2}{EE'} + \frac{c^2}{FF'} = 0.$$

PROPOSITION IV.

1. **Dem.**—Let O, O′ be the centres. Join OA, O′A′, and let fall ⊥ˢ OC, O′C′ on AA′. From T let fall a ⊥ TD on AA′.

Now in the △ˢ ACO, ADT we have the ∠ˢ ACO, ADT equal, and the ∠ OAT is right (III. xviii.), and is equal to the sum of the ∠ˢ OAC, AOC. Reject OAC, and we have AOC = TAD;



∴ the △ˢ OAC, ADT are equiangular; hence (iv.) OA : AC :: AT : TD; alternation, OA : AT :: AC : TD. Similarly, O′A′ : A′T :: A′C′ : TD; but AC = A′C′, since AB, A′B′ are

equal, and (III. III.) are bisected in C, C′; hence the ratio of AC : TD is equal to the ratio of A′C′ : TD; ∴ AO : AT : : A′O′ : A′T; alternation, AO : A′O′ : : AT : A′T.

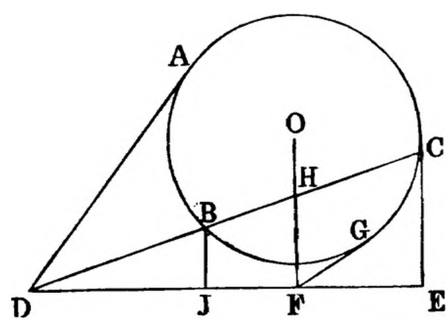
2. **Dem.**—As in the last exercise, let AB, A′B′ be the chords, having a given ratio. The construction will be as before, and we will have the $\triangle^s$ ATD, ACO equiangular; then (IV.) AT : TD : : AO : AC, and TD : TA′ : : C′A′ : A′O′. Multiply together, and we get AT : TA′ : : AO . C′A′ : AC . A′O′; hence

$$\frac{AT}{TA'} = \frac{AO}{A'O'} \cdot \frac{C'A'}{AC} = \frac{AO}{A'O'} \cdot \frac{A'B'}{AB}.$$

3. Let ABC be the ⊙, and DE the line.

Sol.—From the centre O let fall a ⊥ OF on DE. Draw FG a tangent. In FO take FH = FG. H is the required point.

Dem.—Through H draw CD, meeting the ⊙ in B, C, and DE in D. From B, C let fall $\perp^s$ BJ, CE on DE, and draw AD a tangent to the circle.



Now ("Sequel," Book III., Prop. XXI.) $DA^2 = DF^2 + FG^2 = DF^2 + FH^2 = DH^2$; but DA^2 = CD . DB (III. XXXVI.); ∴ CD . DB = DH^2; ∴ CD : DH : : DH : DB.

Again, CD : DH : : CE : HF (IV.), and DH : DB : : HF : BJ; ∴ CE : HF : : HF : BJ; that is, CE . BJ = HF^2.

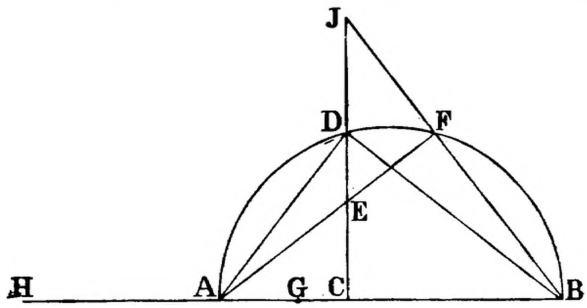
4. Let the given ratio be the ratio of AC : AG.

Sol.—Join BD, and produce GA to H, so that GH . AH = BD^2 (II. VI., Ex. 2). Place a chord BF in ADB = AH (IV. I.). Join AF. AF is the required line.

Dem.—Join AD, and produce CD, BF to meet in J.

Now the ∠ AFB is right (III. XXXI.), ∴ AFJ is right, and ACJ is right; hence ACFJ is a cyclic quadrilateral; hence JB . BF = AB . BC; but AB . BC = BD^2, that is = to GH . AH; ∴ JB . BF = GH . AH; but BF = AH (const.); ∴ JB = GH, and JF = AG.

Again, AC : CE : : JF : EF (IV.); alternation, AC : JF : : CE



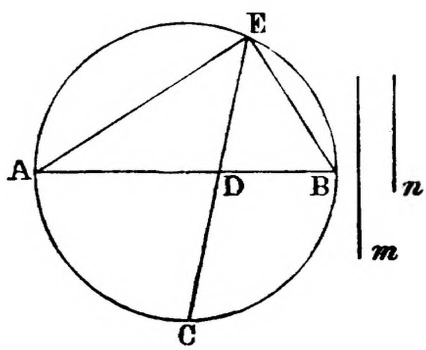
: EF; but JF = AG. Hence AC : AG : : CE : EF.

PROPOSITION X.

3. See "Sequel," Book VI., Prop. v., Sect. iii.

4. Let the sum of the squares of the lines be equal to the squares on AB, and their ratio that of $m : n$.

Sol.—On AB as diameter describe a ⊙ ABC. Divide AB in D, so that AD : DB : : $m : n$ (Ex. 1). Bisect the arc ACB in C



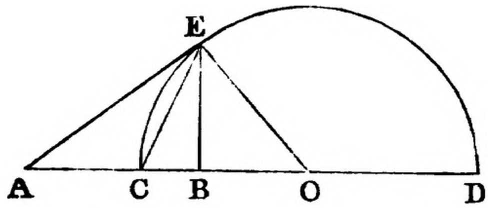
Join CD, and produce it to meet the circumference in E. Join AE, BE. AE, BE are the required lines.

Dem.—$AB^2 = AE^2 + BE^2$, and (III. XXVII.) the ∠ AEB is bisected; hence AE : EB : : AD : DB; but AD : DB : : $m : n$. Hence AE : EB : : $m : n$.

5. Let the difference of the squares of the lines be equal to AB^2, and their ratio that of $m : n$.

Sol.—Divide AB internally and externally in C and D, in the ratio of $m : n$. On CD as diameter describe a semicircle. Let

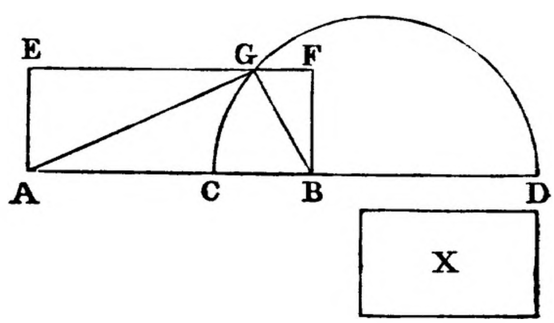
O be its centre. Erect BE ⊥ to AD, and join AE. AE and BE are the required lines.



Dem.—Join OE, CE. Now (I. XLVII.) $AE^2 - BE^2 = AB^2$. And because AB is divided harmonically in C and D, and CD is bisected in O, OB, OC, OA are in geometrical progression (Book V., Ex. 9). Hence $OA . OB = OC^2 = OE^2$, ∴ the ∠ AEO is right, ∴ the ∠ OAE = BEO; but ECO = CEO (I. V.). Hence (I. XXXII.) the ∠ AEC = CEB; ∴ (III.) AE : EB :: AC : CB; that is, :: $m : n$.

6. (1) Let AB be the base; $m : n$ the ratio of the sides, and the rectangle X the area.

Sol.—Divide AB internally and externally in C and D, in the



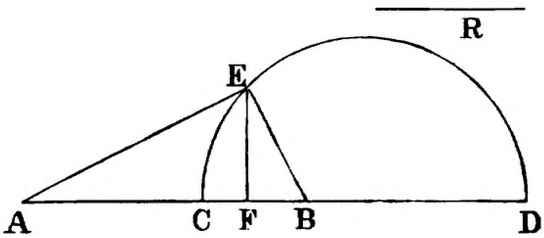
ratio $m : n$ (Ex. 1). On CD as diameter describe a ⊙; to AB apply a parallelogram AF, whose area is 2 X. Let its side EF cut the ⊙ in G. Join AG, BG. AGB is the Δ required.

Dem.—AG : GB :: AC : CB (Dem. of Ex. 4), that is as $m : n$, and the parallelogram AF = 2 AGB; but AF = 2 X; ∴ AGB = X.

(2) Let AB be the base; $m : n$ the ratio of the sides, and R^2 the difference of the squares of the sides.

Sol.—Divide AB as in (1). On CD as diameter describe a ⊙. Divide AB in F, so that $AF^2 - BF^2 = R^2$ ("Sequel," Book I.,

Prop. IX.). Erect FE ⊥ to AD, cutting the ⊙ in E. Join AE, BE. AEB is the △ required.



Dem.—AE : EB : : AC : CB, that is as $m : n$, and $AE^2 - EB^2 = AF^2 - FB^2 = R^2$.

(3) Let AB be the base; $m : n$ the ratio of the sides, and $2\,R^2$ the sum of the squares of the sides.

Sol.—Divide AB as in (1), and on CD as diameter describe a ⊙ CDE. Bisect AB in F. Erect FG ⊥ to AD. From A inflect AG on FG, and equal to R. With F as centre, and FG as radius, describe a ⊙, cutting CDE in E. Join AE, BE. AEB is the △ required.

Dem.—Join FE. Now as in (1) AE : BE : : $m : n$, and FG = FE (const.); ∴ $FG^2 = FE^2$; ∴ $AF^2 + FG^2$; that is, $AG^2 = AF^2 + FE^2$, ∴ $2\,AG^2$, that is, $2\,R^2 = 2\,AF^2 + 2\,FE^2$. Hence (II. x., Ex. 2) $AE^2 + BE^2 = 2\,R^2$.

(4) Let AB be the base; $m : n$ the ratio of the sides, and X the vertical ∠.

Sol.—Divide AB as in (1). On CD as diameter describe a ⊙ CDE; and on AB describe a ⊙ AEB, containing an ∠ = X. Join AE, BE. AEB is the △ required.

Dem.—AE : BE : : $m : n$, and the vertical ∠ AEB = X.

(5) Let X be the difference of the base angles.

Sol.—Divide AB as in (1), and on CD describe a ⊙ CDF. Erect CE ⊥ to AD; and at C, in the line CE, make the ∠ ECF = $\frac{1}{2}$ X. Join AF, BF. AFB is the △ required.

Dem.—AF : BF : : $m : n$; and the difference between the ∠s ACF, BCF is equal to 2 ECF = X; but ACF = CBF + CFB, and BCF = CAF + CFA, and CFA = CFB. Hence CBF − CAF = ACF − BCF = X.

PROPOSITION XI.

1. **Dem.**—Join OB, B′C, &c. Now in the $\triangle^s$ OAB, BB′C, we have OA : AB : : B′B : BC, and the right $\angle$ OAB = B′BC ; hence (vi.) the $\triangle^s$ are equiangular, $\therefore$ the $\angle$ ABO = BCB′; hence OB, B′C are parallel. Similarly B′C, C′D are parallel. Now, since the lines AO, BB′, CC′ are parallel, we have (ii., Ex. 1) OB′ : B′C′ : : AB : BC ; and because OB, B′C, C′D are parallel, OB′ : BC : : BC : CD ; hence AB : BC : : BC : CD. In like manner BC : CD : : CD : DE. Hence AB, BC, CD, &c., are in continued proportion.

2. **Dem.**—Because B′M is $\parallel$ to AΩ, the $\triangle^s$ OMB′, OAΩ are equiangular ; $\therefore$ OM : MB′ : : OA : AΩ ; but OM = OA − AM = AB − BB′ = AB − BC, and MB′ = AB, and OA = AB. Hence AB − BC : AB : : AB : AΩ.

PROPOSITION XIII.

1. (Diagram to Prop. viii.).

Sol.—Let AB, BD be the two lines. On AB describe a semicircle. At D erect DC $\perp$ to AB, and meeting the semicircle in C. Join BC. BC is a mean proportional between AB, BD.

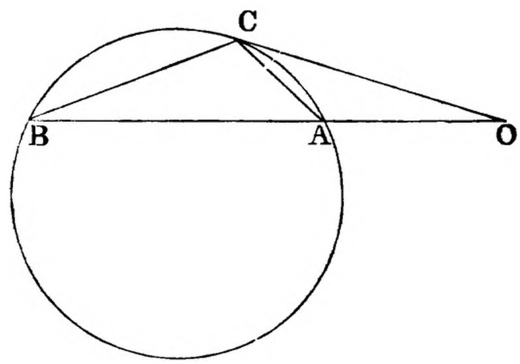
Dem.—Join AC. Now the $\angle^s$ ABC, BCD are equiangular (viii.), $\therefore$ AB : BC : : BC : BD. Hence BC is a mean proportional between AB and BD.

2. **Sol.**—Let O be any point taken within a $\odot$ ABC ; O′ the centre. Join OO′, and produce both ways to meet the circumference in A, B. Through O draw CD $\perp$ to AB. CD is bisected in O (III. iii.). Through O draw any other chord FE. OC is a mean proportional between OF and OE.

Dem.—Join CF, DE. Now, because the $\triangle^s$ OCF, OED are equiangular (III. xxi.), we have (iv.) OF : OC : : OD : OE; but OD = OC ; $\therefore$ OF : OC : : OC : OE. Hence OC is a mean proportional between OF and OE.

3. Let ABC be a $\odot$, O any external point. From B draw a secant BO, and from O draw OC a tangent to the $\odot$. It is

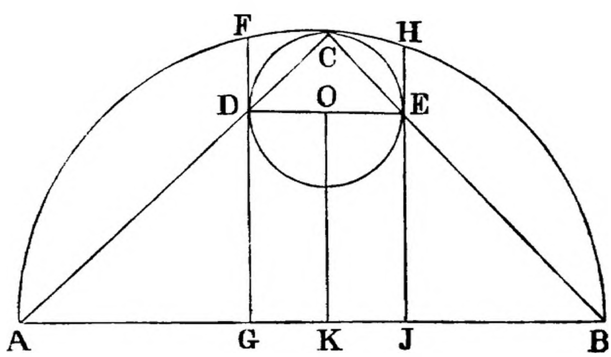
required to prove that OC is a mean proportional between OB and OA.



Dem.—Join AC, BC. Now in the $\triangle^s$ OAC, OBC, we have the $\angle$ OCA = OBC (III. XXXII.), and the $\angle$ BOC common; hence the $\triangle^s$ are equiangular, $\therefore$ BO : OC : : OC : OA. Hence OC is a mean proportional between OB and OA.

4. **Dem.**—Let AB be the chord of the arc. Join AE, AC, CB. Now because the arc AC = BC, the $\angle$ CAB = CBA; but CBA = AEC (III. XXI.); $\therefore$ AEC = CAD, and the $\angle$ ACD is common; $\therefore$ the $\triangle^s$ ACE, ACD are equiangular; $\therefore$ EC : AC : : AC : CD. Hence AC is a mean proportional between CE and CD.

5. Let ACB be a $\odot$ whose diameter is AB; FG, HJ two parallel chords; CDE a $\odot$ touching ACB internally in C; and

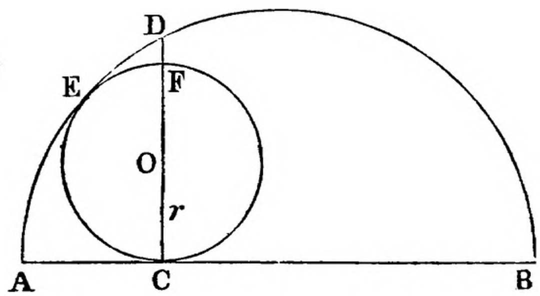


FG, HJ in D, E. From O, the centre of CDE, let fall a $\perp$ OK on AB. It is required to prove that OK is a mean proportional between AG and JB.

Dem.—Join OD, OE, CD, CE. CD, CE produced must

pass through A, B (III., Ex. 51). Now (III. xviii.) the ∠ ODG is right, and DGB is right; ∴ OD is ∥ to AB. Similarly OE is ∥ to AB; ∴ OD, OE are in the same straight line. Again, since the ∠ AGD is right, the ∠s GAD, GDA are equal to a right ∠; and because ACB is right (III. xxxi.), the ∠s CAB, CBA are equal to a right ∠; hence the ∠ GDA = JBE, and the ∠ DGA = EJB; ∴ the △s ADG, JEB are equiangular; hence AG : GD :: EJ : JB; but GD and EJ are each equal to OK; ∴ AG : OK :: OK : JB. Hence OK is a mean proportional between AG and JB.

6. Let ADB be a semicircle whose diameter is AB; CEF a ⊙ touching ADB in E and AB in C. Through O, its centre, draw



the diameter CF, and produce it to meet ADB in D. It is required to prove that CF is a harmonic mean between AC and CB.

Dem.—$AB . r = CD^2$ ("Sequel," III., Prop. v.); but $AC . CB = CD^2$, ∴ $AB . r = AC . CB$, ∴ $r = \frac{AC . CB}{AC + CB}$; ∴ $2r = \frac{2 AC . CB}{AC + CB}$. Hence (V., Miscellaneous, Ex. 11.) $2r$, that is CF, is a harmonic mean between AC and CB.

7. Let ACB be a ⊙ whose diameter is AB; FG, HJ, two parallel chords meeting the ⊙ in F, H, and the diameter in G, J. Describe a ⊙ CDE touching ACB externally in C, and GF, JH produced in D, E. From O, its centre, let fall a ⊥ OK on AB. It is required to prove that OK is a mean proportional between AJ and GB.

The proof is the same as in Ex. 5.

PROPOSITION XVII.

2. **Dem.**—Describe a ⊙ about the △. Produce AC to G, and bisect the external ∠ BCG by CD′, meeting AB produced in D′. Produce D′C to meet the ⊙ in F, and join AF. Now the ∠ BCD′ = GCD′, and GCD′ = FCA, ∴ BCD′ = FCA; and since the ∠s CBD′, CBA are together equal to two right ∠s, and the ∠s CFA, CBA are equal to two right ∠s, the ∠ CBD′ = CFA; ∴ the △s AFC, BCD′ are equiangular; ∴ AC : CF :: D′C : CB (IV.); hence AC . CB = D′C . CF. Again AD′ . D′B = FD′ . D′C; but FD′ . D′C = FC . CD′ + CD′^{2} (II. III.) = AC . CB + CD′^{2}. Hence AD′ . D′B′ − CD′^{2} = AC . CB.

4. **Dem.**—Let O′ be the centre of the escribed ⊙, touching AB externally, and the other sides produced. Join O′C, cutting the circumscribed ⊙ in E. Through E draw EF, the diameter of the circumscribed ⊙. Join O′B, EB, FB, O′G, G being the point where CB produced touches the escribed ⊙.

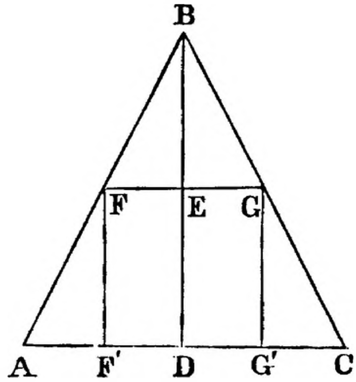
Now the ∠s O′GC, EBF are equal, each being right, and the ∠ O′CG = EFB (III. XXI.); ∴ the △s O′CG, BFE are equiangular; hence (IV.) FE : EB :: O′C : O′G, and EB = EO′ (Dem. of IV., Ex. 19); hence FE : EO′ :: O′C : O′G, ∴ FE . O′G = EO′ . O′C; that is, the rectangle contained by the diameter of the circumscribed ⊙, and the radius of the escribed ⊙, is equal to the rectangle contained by the segments of any chord of the circumscribed ⊙ passing through the centre of the escribed ⊙.

7. **Dem.**—Produce AD to meet the circumference in G; then (Ex. 6) we have AB . AE + AC . AF = AG . AD; but AG . AD = GD . DA + DA^2 (II. III.), and GD . DA = BD . DC (III. XXXV.). Hence AB . AE + AC . AF = BD . DC + DA^2.

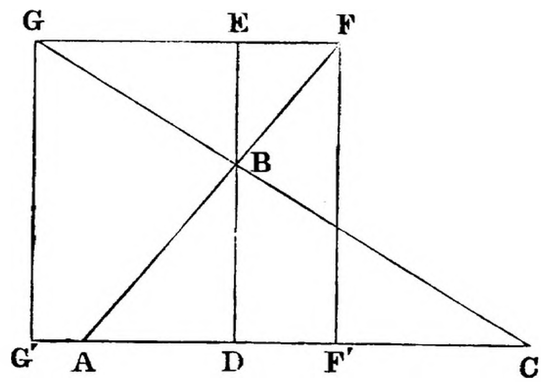
9. **Dem.**—Let ABC be the △, and FGG′F′ the inscribed square. From B let fall a ⊥ BD on AC, cutting the side FG of the square in E.

Now AC : FG :: AB : FB (IV.); but AB : FB :: BD : BE (IV.); ∴ AC : FG :: BD : BE. Hence, putting *b* for base, *p* for

$\perp$, and s for side of square, we have $b : s :: p : p - s$; $\therefore$ $bp - bs = sp$. Hence $bp = (b + p) s$.



10. Dem.—Let ABC be the $\triangle$, and FGF′G′ the escribed



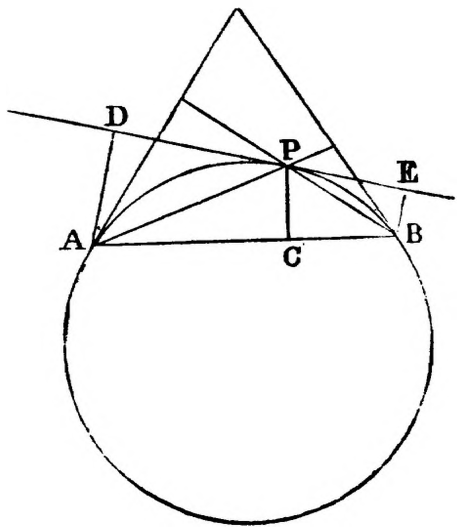
square. From B let fall a $\perp$ BD on AC, and produce it to meet FG in E.

Now AC : FG : : AB : BF (IV.) ; but AB : BF : : BD : BE (IV.) ; $\therefore$ AC : FG :: BD : BE; that is, putting s' for the side of the square, $b : s' :: p : s' - p$. Hence $bs' - bp = s'p$; $\therefore$ $bp = s' (b - p)$.

11. From P let fall a $\perp$ PC on the chord AB, and from A, B let fall $\perp^s$ AD, BE on DE, the tangent at P. It is required to prove that $CP^2 = AD \cdot BE$.

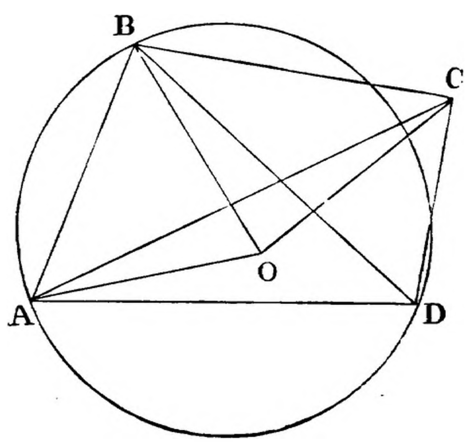
Dem.—Join AP, BP. Now in the $\triangle^s$ APD, BPC, the $\angle$ APD = BPC (III. XXXII.), and the $\angle$ ADP = BCP, $\therefore$ the $\triangle^s$ are equiangular; hence (IV.) AP : AD : : BP : PC; alternation, AP : BP : : AD : PC. In like manner for the $\triangle^s$ APC, BPE,

we have AP : BP : : PC : BE, ∴ AD : PC : : PC : BE. Hence CP^2 = AD . BE.



12. **Dem.**—In the △s AOD, BOC, the ∠ AOD = BOC, and the ∠ OAD = OBC (III. xxi.) ; hence (iv.) AD : AO : : BC : BO ; alternation, AD : BC : : AO : BO. Multiplying each by AB, we get AD . AB : AB . BC : : AO : BO. Similarly AB . BC : BC . CD : : BO : CO, &c. Hence the four rectangles are proportional to the four lines.

14. **Dem.**—Draw the diagonals AC, BD. Make the ∠ ABO



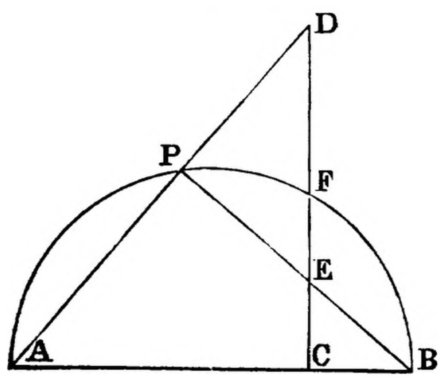
= DBC, and BAO = BDC. Join OC.

Now the $\triangle^s$ ABO, DBC are equiangular, $\therefore$ AB : AO : : BD : DC, $\therefore$ AB . CD = AO . BD. Again, since AB : BO : : BD : BC; alternation, AB : BD : : BO : BC; and since the $\angle$ ABO = DBC, the $\angle$ ABD = OBC; hence (IV.) the $\triangle^s$ ABD, OBC are equiangular; $\therefore$ (IV.) AD : BD : : OC : BC; hence AD . BC = BD . OC. Now we have proved AB . CD = AO . BD, AD . BC = OC . BD, and AC . BD = AC . BD; hence the three rectangles are proportional to the sides AO, OC, AC of the $\triangle$ AOC; and since the $\triangle^s$ AOB, CDB have been shown to be equiangular, the $\angle$ AOB = BCD; and because the $\triangle^s$ BOC, ABD are equiangular, the $\angle$ COB = BAD. Hence the $\angle$ AOC is equal to the sum of the $\angle^s$ BAD, BCD.

15. Let ABCD be a cyclic quadrilateral; AC, BD its diagonals. At P, any point in the circumference of the circumscribed $\odot$, draw a tangent to the $\odot$, and let fall $\perp^s$ PE, PF, PG, PL on AB, BD, AC, CD. It is required to prove that PF . PG = PE . PL.

Dem.—From A, B, C, D let fall $\perp^s$ AH, BI, CJ, DK on the tangent at P. Now PF^2 = BI . DK (Ex. 11), and PG^2 = AH . CJ; $\therefore$ $PF^2 . PG^2$ = BI . DK . AH . CJ. In like manner $PE^2 . PL^2$ = BI . AH . DK . CJ; $\therefore$ $PF^2 . PG^2 = PE^2 . PL^2$. Hence PF . PG = PE . PL.

16. **Dem.**—The $\angle$ APB is right (III. XXXI.); $\therefore$ DPE is right, and equal to ECB, and PED = CEB, $\therefore$ PDE = CBE. Now since PDE = CBE, and ACD = ECB, the $\triangle^s$ ADC, EBC



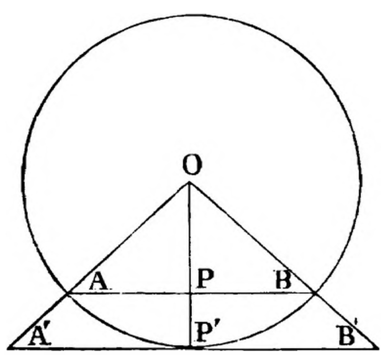
are equiangular; hence AC : CD : : CE : CB (IV.), $\therefore$ AC . CB = CD . CE; but AC . CB = CF^2 (XVII.); $\therefore$ CD . CE = CF^2. Hence CF is a mean proportional between CD and CE.

PROPOSITION XIX.

1. Let ABC, DEF be the two $\triangle^s$. Now AB $= \frac{3}{2}$ DE (hyp.), $\therefore$ AB : DE : : 3 : 2 ; $\therefore$ $AB^2 : DE^2$: : 9 : 4 ; but ABC : DEF : : $AB^2 : DE^2$ (XIX.) Hence the $\triangle$ ABC : DEF : : 9 : 4.

2. Let AB be a side of the inscribed polygon, O the centre of the $\odot$. Join OA, OB, and bisect the $\angle$ AOB by OP′, meeting AB in P. Through P′ draw a tangent to the $\odot$, and produce OA, OB to meet it; then evidently A′B′ is a side of the circumscribed polygon.

Now, if each of the polygons have n sides, and we denote their areas by π and π', we have the $\triangle$ AOB $= \frac{\pi}{n}$, and A′OB′ $= \frac{\pi'}{n}$; hence AOB : A′OB′ : : $\pi : \pi'$; but (XIX.) AOB : A′OB′ : : AO^2

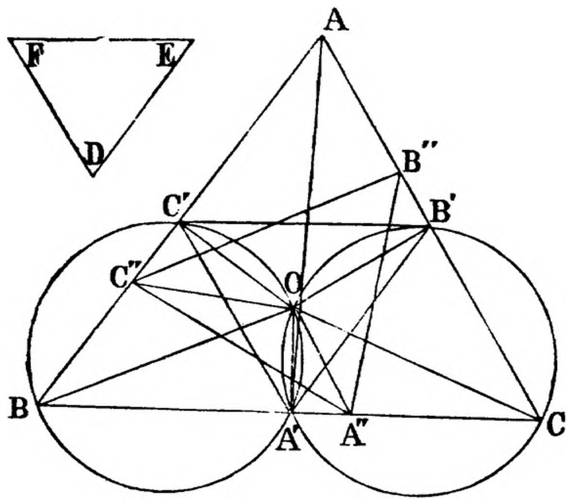


: $A'O^2$; that is, : : $OP^2 : OP'^2$ (IV.), or : : $OP^2 : OA^2$; hence $\pi : \pi'$: : $OP^2 : OA^2$; $\therefore \pi' - \pi : \pi$: : $AP^2 : OA^2$; that is, as $4\ AP^2 : 4\ OA^2$; that is, as AB^2 is to the square of the diameter ; but π is less than the square of the diameter (IV., Ex. 37). Hence $\pi' - \pi$ is less than AB^2.

PROPOSITION XX.

4. **Dem.**—Let AB, BC, CA be three given lines in the form of a $\triangle$. Inscribe in ABC a $\triangle$ A′B′C′ similar to the $\triangle$ FDE. About the $\triangle^s$ A′BC′, A′B′C describe $\odot^s$ intersecting in O ; then

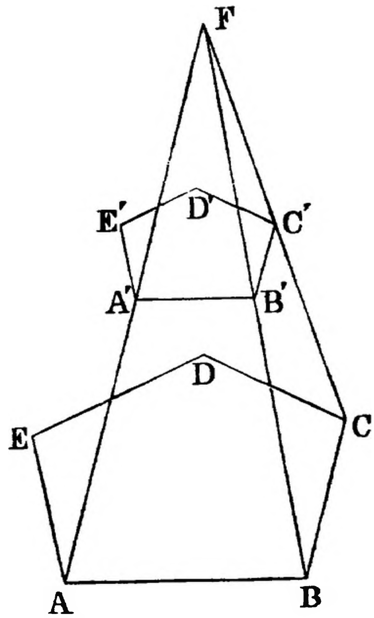
the ⊙ about AB′C′ will pass through O (III., Ex. 28). Join OA′, OB, OC, OB′, OC′, AA′. Now (III. XXI.) the ∠ BOA′ = BC′A′, and COA′ = CB′A′; ∴ the ∠ BOC is equal to the sum of the ∠ˢ BC′A′, CB′A′; but BC′A′ = BAA′ + AA′C′, and CB′A′ = CAA′ + AA′B′; ∴ the ∠ BOC = C′AB′ + C′A′B′; but C′A′B′ = FDE; hence BOC = C′AB′ + FDE; but the ∠ FDE is given, and C′AB′ is given; ∴ the ∠ BOC is given, and the base BC is given;



hence the ⊙ described about the △ BOC is given in position. Similarly, the ⊙ˢ about the △ˢ AOB, AOC are given in position; hence O is a given point. Hence, if we inscribe another △ A″B″C″ similar to FDE in ABC, the ⊙ˢ described about the △ˢ A″BC″, B″CA″, C″AB″ will co-intersect in O, and if we join the angular points to O, the ∠ˢ OC″A″, OA″C″ will be equal to the ∠ˢ OBA′, OBC′; that is, equal to the ∠ˢ OC′A′, OA′C; hence the △ˢ OC′A′, OC″A″ are equiangular, and therefore (Ex. 2) O is the centre of similitude of the △ˢ A′B′C′, A″B″C″.

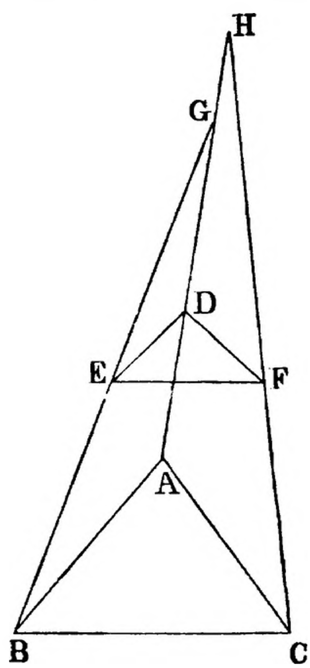
5. Let ABCDE, A′B′C′D′E′ be two similar figures, having the sides AB, BC parallel to the sides A′B′, B′C′. It is required to prove that the other homologous sides are parallel.

Dem.—Join AA′, BB′, and produce them to meet in F. Now the ∠ BAF = B′A′F (I. XXIX.); but since the figures are similar, the ∠ BAE = B′A′E′; hence the ∠ FAE = FA′E′, and therefore



the line AE is parallel to A′E′. Similarly, it can be shown that the other homologous sides are parallel.

6. Let ABC, DEF be the homothetic figures. Join BE, AD,



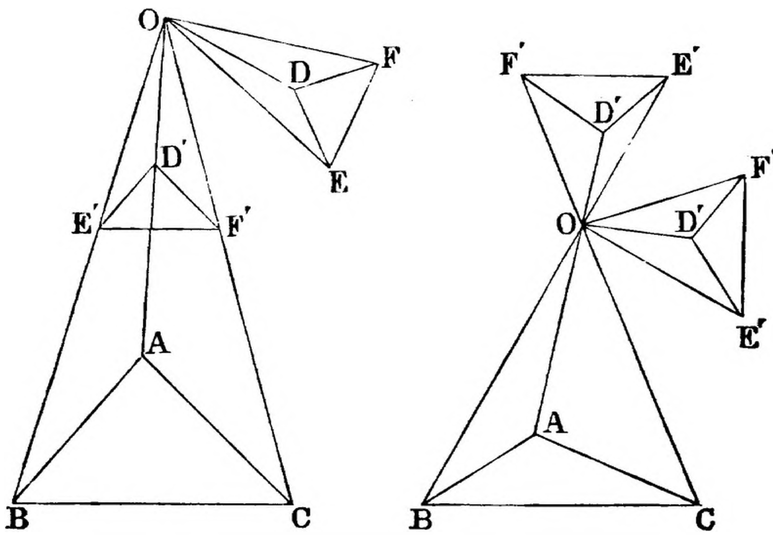
and produce them to meet in G. Join CF. It is required to prove that CF produced will pass through G.

Dem.—If not, let it pass through H. Produce AG to H.

Now the ∠ GED = GBA (I. xxix.), and the ∠ GDE = GAB; hence (iv.) AG : AB : : DG : DE; but AB : AC : : DE : DF; ∴ AG : AC : : DG : DF; alternation, AG : DG : : AC : DF.

Again, since the $\triangle^s$ HAC, HDF are equiangular, we have AH : AC : : DH : DF; alternation, AH : DH : : AC : DF; ∴ AH : DH : : AG : DG; hence (V. xvii.) AD : DH : : AD : DG; and therefore DH = DG, which is absurd. Hence CF produced must pass through G.

7. **Dem.**—Let ABC, DEF be the two similar figures; O their centre of similitude. Join OA, OB, OC, OD, OE, OF. From OA, OB, OC cut off OD′, OE′, OF′ equal respectively to OD, OE, OF, and join D′E′, D′F′, E′F′. Now since OD′ = OD, OE′ = OE, and the ∠ D′OE′ = DOE (hyp.), ∴ DE = D′E′,

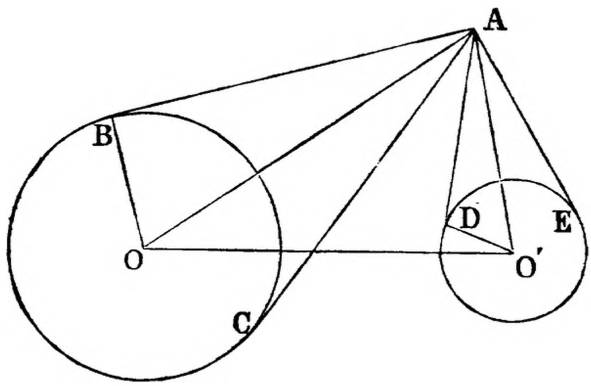


and the ∠ OED = OE′D′; but OED = OBA (hyp.); ∴ OE′D′ = OBA; ∴ D′E′ and AB are parallel. Similarly, D′F is ∥ to AC and equal to DF, and E′F is equal to EF and ∥ to BC; hence the figure DEF may be turned round O so as to take up the position D′E′F′. In like manner the figure may be turned round in the opposite direction, as in the second diagram.

10. **Dem.**—Let O, O′ be the centres of the ⊙^{s}, and A one of their centres of similitude. Join OO′, and from A draw AB, AC, AD, AE tangents to the ⊙^{s}. Join OA, OB, O′A, O′D.

Now since A is a centre of similitude, the ∠ BAC = DAE; therefore their halves are equal; that is, the ∠ BAO = DAO′,

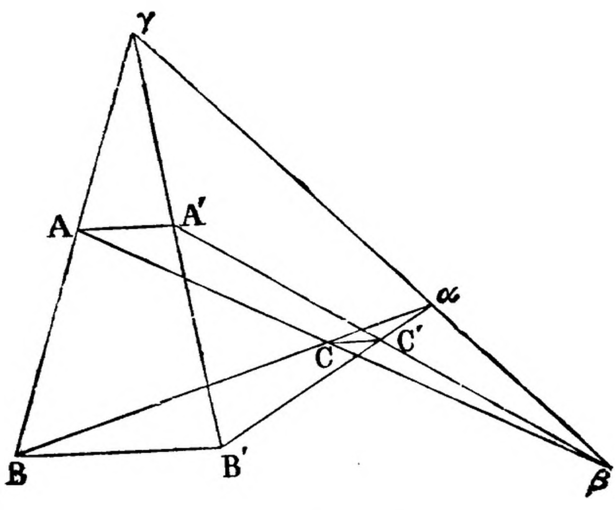
and the right $\angle^s$ ABO, ADO′ are equal; ∴ the Δ^s ABO, ADO′ are equiangular; hence AO : OB : : AO′ : O′D; alternation, AO : AO′ : : OB : O′D; but the ratio OB : O′D is given, since



OB and O′D are given lines; hence the ratio AO : A′O is given. Now in the Δ OAO′ we have the base OO′ given, and the ratio of the sides. Therefore (III., Ex. 6) the locus of A is a circle.

PROPOSITION XXI.

1. **Dem.**—Let AA′, BB′, CC′ be corresponding sides of the similar rectilineal figures; then since the figures are homothetic,



these sides are parallel. Join BA, B′A′, and produce to meet in γ; then because AA′, BB′ are corresponding sides of the

homothetic figures, γ will be their centre of similitude. In like manner, if we join BC, B′C′, and produce to meet in α; AC, A′C′ to meet in β; α and β will be centres of similitude.

Now (IV.) $\frac{B\gamma}{\gamma A} = \frac{BB'}{AA'}$. Similarly, $\frac{C\alpha}{\alpha B} = \frac{CC'}{BB'}$, and $\frac{A\beta}{\beta C} = \frac{AA'}{CC'}$; but the product of $\frac{BB'}{AA'}$, $\frac{CC'}{BB'}$, $\frac{AA'}{CC'}$ is unity; $\therefore$ the product of $\frac{B\gamma}{\gamma A}$, $\frac{C\alpha}{\alpha B}$, $\frac{A\beta}{\beta C}$ is unity. And hence ("Sequel," Book VI., Prop. IV., Cor. 1, p. 69), the points α, β, γ are collinear.

PROPOSITION XXIII.

1. **Dem.**—Let ABC, DEF be the $\triangle$s having the $\angle$ ABC = DEF. Complete the parallelograms ABCG, DEFH. Now the $\triangle$ ABC : DEF : : ABCG : DEFH; but ABCG : DEFH : : AB . BC : DE . EF (XXIII.). Hence ABC : DEF : : AB . BC : DE . EF.

2. Let ABCD, EFGH be two quadrilaterals whose diagonals AC, BD; EG, FH intersect in I, J, making the $\angle$ CIB = GJF. It is required to prove that ABCD : EFGH : : AC . BD : EG . FH.

Dem.—The area of ABCD is equal to the area of a $\triangle$ having two sides equal to AC, BD, and the contained $\angle$ equal to CIB (I. XXXIV., Ex. 7); and EFGH is equal to a $\triangle$ having two sides equal to EG, FH, and the contained $\angle$ equal to GJF; but (Ex. 1) those $\triangle$s are to one another as AC . BD : EG . FH. Hence ABCD : EFGH : : AC . BD : EG . FH.

PROPOSITION XXX.

1. Let ABC be a right-angled $\triangle$ whose sides are in continued proportion; that is, having AB : BC : : BC : CA. From C let fall a $\perp$ CD on AB. It is required to prove that AB is divided in extreme and mean ratio in D.

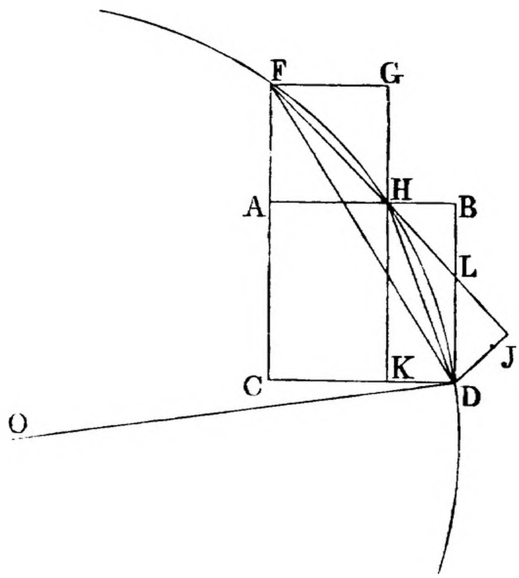
Dem.—Because AB : BC : : BC : CA, AB . AC = BC^2. Again (I. XLVII., Ex. 1), AB . BD = BC^2; $\therefore$ AC = BD, and AB . AD = AC^2; $\therefore$ AB . AD = BD^2. Hence AB is divided in extreme and mean ratio in D.

2. See Demonstration of last Exercise.

3. Join FD, and describe a ⊙ about the △ FHD. Let O be its centre. Join DO, and produce it to meet the circumference in I. It is required to prove that $DI^2 = 6\,FD^2$.

Dem.—Join IF. Produce FH, and let fall a ⊥ DJ on it.

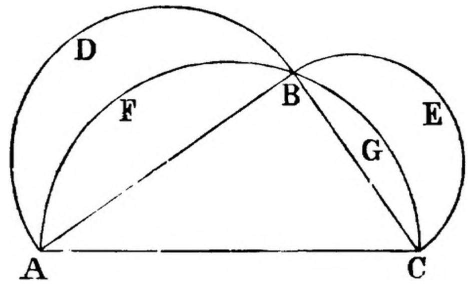
Now since AG is a square, AF = AH; ∴ the ∠ AHF = AFH, and FAH is a right ∠; ∴ AHF is half a right ∠; ∴ BHL is half a right ∠, and HBL is a right ∠; ∴ HLB is half a right ∠, and BH = BL, and the ∠ DLJ = BHL; ∴ DLJ is half a right ∠, and DJL is a right ∠; ∴ JDL is half a right ∠, and JL = JD; ∴ $JL^2 = JD^2$, and $DL^2 = 2\,DJ^2$.



Again, since AB = DB, and BH = BL, ∴ DL = AH; but AB is divided in extreme and mean ratio in H, ∴ BD is divided in extreme and mean ratio in L; and hence (II. xi., Ex. 4) $BD^2 + BL^2 = 3\,DL^2 = 6\,DJ^2$; hence $BD^2 + BH^2$; that is, $DH^2 = 6\,DJ^2$. Again (III. xxii.), the ∠s FHD, FID are together equal to two right ∠s, and the ∠s FHD, DHJ are equal to two right ∠s; ∴ the ∠ FID = DHJ, and the right ∠ IFD = HJD; ∴ the △s IFD, DHJ are equiangular; ∴ ID : DF : : DH : DJ; ∴ $ID^2 : DF^2 : : DH^2 : DJ^2$; but $DH^2 = 6\,DJ^2$. Hence $ID^2 = 6\,DF^2$.

PROPOSITION XXXI.

1. **Dem.**—Let ABC be the semicircle, of which AB, CB are supplemental chords. On AB, CB describe semicircles ADB, BEC. Now (XXXI.) the semicircle ABC is equal to the sum of



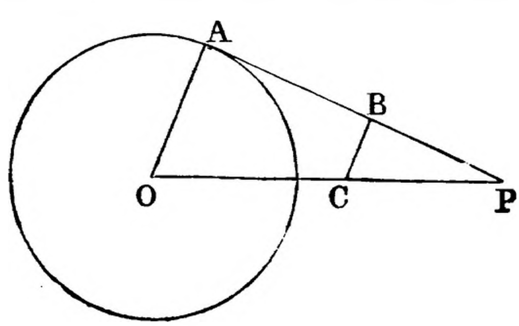
the semicircles ADB, BEC. Take away the common segments AFB, BGC, and we have the △ ABC equal to the sum of the crescents ADBF, BECG.

Exercises on Book VI.

1. Let ACB be a fixed △, DE a parallel to AB. Draw the diagonals AE, BD, intersecting in O. Join CO, and produce it to meet AB in H. It is required to prove that CH bisects AB.

Dem.—Through O draw FG parallel to AB. Now (II.) AE : EO : : BD : DO; but, by similar △s, AE : EO : : AB : OG, and BD : DO : : AB : OF ; hence AB : OG : : AB : OF, and therefore OG = OF. Now ACB is a △ ; and FG, a parallel to the base, is bisected by CO. Hence AB is bisected by CO.

2. Let O be the centre of the ⊙, and P the given point.

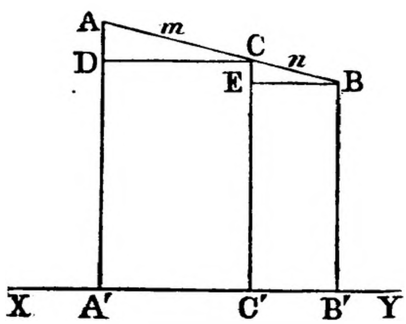


From P draw PA to any point A in the ⊙. Divide AP at B in a given ratio. It is required to find the locus of B.

Sol.—Join OP, OA, and draw BC ∥ to AO.

Now PB : BA : : PC : CO (II.); but the ratio PB : BA is given; ∴ PC : CO is given, and therefore C is a given point. Again, by similar $\triangle^s$, we have PA : AO : : PB : BC; alternation, PA : PB : : AO : BC; but the ratio PA : PB is given, ∴ AO : BC is given; but AO is given; ∴ BC is given, and the point C is given. Hence the locus of B is a ⊙, having C as centre and BC as radius.

3. **Dem.**—Through B, C draw BE, CD ∥ to XY.

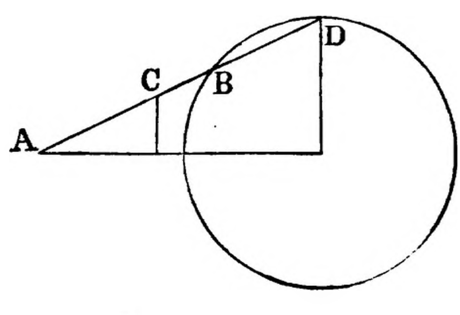


Now, by similar $\triangle^s$, AC : AD : : CB : CE; alternation, AC : CB : : AD : CE, ∴ AD : CE : : $m : n$; but AD = AA′ − A′D = AA′ − CC′, and CE = CC′ − C′E = CC′ − BB; hence AA′ − CC′ : CC′ − BB′ : : $m : n$; ∴ n AA′ − n CC′ = m CC′ − m BB′; and hence m BB′ + n AA′ = $(m + n)$ CC′.

4. See "Sequel," Book VI., Prop. II., Section 1.

5. See "Sequel," Book VI., Prop. IV., Section 1.

6. **Dem.**—Let the rectangle AB . AC = k^2. Produce AB to



meet the circumference in D. Now, if t denote the tangent drawn rom A to the ⊙ (III. XXXVI.), AB . AD = t^2; ∴ AB . AD : AB . AC : : $t^2 : k^2$; that is, AD : AC : : $t^2 : k^2$; but the ratio $t^2 : k^2$ is given, ∴ AD : AC in a given ratio, and hence (Ex. 2) he locus of C is a circle.

7. **Dem.**—Join O, the centre of the inscribed ⊙, to the points F, G, H, where the sides AB, AC, BC touch the ⊙. Join OC.

Now since AF = AG, BF = BH, and CG = CH, ∴ AB − AC = BF − CG = BH − CH = 2 DH. Again, $AB^2 - AC^2 = BE^2 - EC^2$ (I. XLVII.); that is (AB + AC) (AB − AC) = (BE + EC) (BE − EC); ∴ (AB + AC) 2 DH = BC . 2 DE; hence (AB + AC) : BC : : DE : DH. Again (III.) AB : AC : : BL : LC; ∴ (AB + AC) : AC : : BC : LC; ∴ (AB + AC) : BC : : AC : LC. Again, AC : LC : : AO : OL (III.); but AO : OL : : HE : HL (II.), ∴ AC : LC : : HE : HL; hence (AB + AC) : BC : : HE : HL; that is, DE : DH : : HE : HL; and hence DE . HL = HE . HD.

8. **Dem.**—Let O′ be the centre of the escribed ⊙, touching BC produced in K. Now (AB + AC) : BC : : AC : LC (Ex. 7); that is, as AO : OL, ∴ (AB + AC + BC) : BC : : AL : OL : : LE : LH; ∴ 2 BK : 2 BD : : LE : LH; hence LH . BK = BD . LE.

9. See Book VI., Prop. XVII., Exs. 3, 4.

10. **Dem.**—From Ex. 9 we have $d^2 = R^2 - 2Rr$; $d'^2 = R^2 + 2Rr'$; $d''^2 = R^2 + 2Rr''$, and $d'''^2 = R^2 + 2Rr'''$; ∴ $d^2 + d'^2 + d''^2 + d'''^2 = 4R^2 + 2R(r' + r'' + r''' - r)$; but (Book III., Ex. 19) $(r' + r'' + r''' - r) = 4R$. Hence $d^2 + d'^2 + d''^2 + d'''^2 = 4R^2 + 2R . 4R = 12R^2$.

11. (1) **Dem.**—Let the sides of the △ be denoted by a, b, c.

Now (IV. IV., Ex. 9) $rs = \Delta$; ∴ $s = \frac{\Delta}{r}$. Again, $ap' = 2\Delta$ (II. I., Cor. 1); ∴ $a = \frac{2\Delta}{p'}$. Similarly, $b = \frac{2\Delta}{p''}$, and $c = \frac{2\Delta}{p'''}$; ∴ $(a + b + c)$, or $2s = \frac{2\Delta}{p'} + \frac{2\Delta}{p''} + \frac{2\Delta}{p'''}$, ∴ $s = \frac{\Delta}{p'} + \frac{\Delta}{p''} + \frac{\Delta}{p'''}$; ∴ $\frac{\Delta}{r} = \frac{\Delta}{p'} + \frac{\Delta}{p''} + \frac{\Delta}{p'''}$; and hence

$$\frac{1}{r} = \frac{1}{p'} + \frac{1}{p''} + \frac{1}{p'''}.$$

(2) $(s - a)r' = \Delta$ (IV. IV., Ex. 10); ∴ $(s - a) = \frac{\Delta}{r'}$. Again, from (1) we have $(b + c - a) = \frac{2\Delta}{p''} + \frac{2\Delta}{p'''} - \frac{2\Delta}{p'}$; but $(b + c - a)$

$= 2(s-a)$; $\therefore (s-a) = \frac{\Delta}{p''} + \frac{\Delta}{p'''} - \frac{\Delta}{p'}$; that is, $\frac{\Delta}{r'} = \frac{\Delta}{p''} + \frac{\Delta}{p'''}$ $- \frac{\Delta}{p'}$. Hence $\frac{1}{r'} = \frac{1}{p''} + \frac{1}{p'''} - \frac{1}{p'}$.

(3) Subtract (2) from (1), and we get $\frac{2}{p'} = \frac{1}{r} - \frac{1}{r'}$.

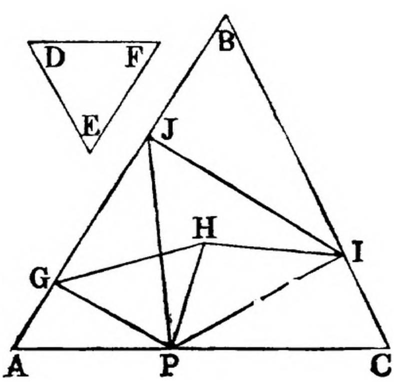
(4) Interchange in (2), and we have $\frac{1}{p'''} + \frac{1}{p'} - \frac{1}{p''} = \frac{1}{r''}$; interchange again, and $\frac{1}{p'} + \frac{1}{p''} - \frac{1}{p'''} = \frac{1}{r'''}$. Add, and we get

$$\frac{2}{p'} = \frac{1}{r''} + \frac{1}{r'''}.$$

12. Let ABC be a given △, and P a given point in one of the sides. It is required to inscribe in ABC a △ equiangular to DEF, and having one of its angular points at P.

Sol.—From P let fall a ⊥ PG on AB. Make the ∠ PGH = EDF, and GPH = DEF. Erect HI ⊥ to PH, meeting BC in I; join PI, and make the ∠ JPI = GPH, and join IJ. JPI is the △ required.

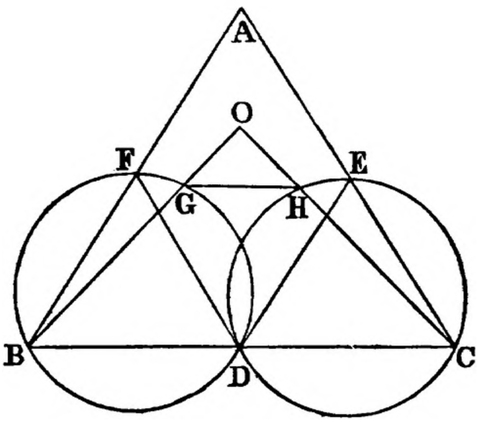
Dem.—Because the ∠ GPH = JPI, ∴ GPJ = HPI, and the right



∠ PGJ = PHI; hence the △s PGJ, PHI are equiangular; ∴ GP : PJ : : HP : PI; alternation, GP : HP : : PJ : PI, and the ∠ GPH = JPI; hence (VI.) the △s GPH, JPI are equiangular; but GPH, DEF are equiangular. Hence JPI is equiangular to DEF, and it has one of its angles at the given point P.

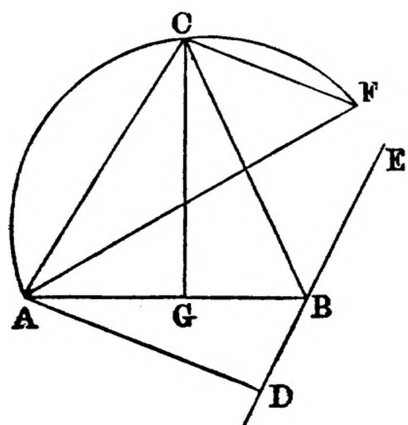
13. Let ABC be a given △, and D, E, F three fixed points in its sides. It is required to prove that the locus of any point O in its plane is a circle.

Dem.—Join OB, OC, DF, DE. Describe ⊙s about the △s DBF, DCE, cutting OB, OC in G, H. Join GH. Now since



the points D, F are given, the line DF is given, and the ∠ DBF is given (hyp.); hence the ⊙ about DBF is given, and the ∠ DBO is given by the given conditions; hence the arc DG is given, and therefore G is a given point. In like manner H is a given point; ∴ the line GH is given, and the ∠ GOH is given. Hence the locus of O is a circle.

14. (1) **Dem.**—Let the point B move along DE. From A let fall a ⊥ AD on DE. Draw AF, making the ∠ DAF = CAB.



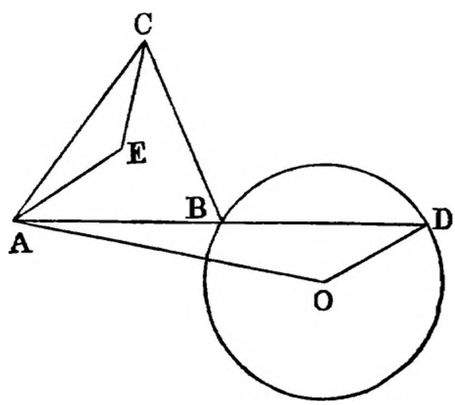
From C draw CF ⊥ to AC, and let fall a ⊥ CG on AB. Now because the ∠ CAG is given, and the ∠ AGC is a right ∠, the △ ACG is given in species; therefore the ratio AC : CG is given;

hence the ratio AC . AB : CG . AB is given; but CG . AB is given, therefore AC . AB is given.

Again, since the ∠ DAF = BAC, ∴ DAB = CAF, and the right ∠ ADB = ACF; therefore the △ˢ DAB, CAF are equiangular; hence AD : AB : : AC : AF, ∴ AB . AC = AD . AF; but AB . AC is given, ∴ AD . AF is given, and AD is given, ∴ AF is given; and since the ∠ DAF is given, ∴ AF is given in position, and the ∠ ACF is right. Hence the locus of C is a circle.

(2) Let the point B move along a ⊙. Produce AB to meet the circumference in D. Let O be the centre. Join OA, OD. Make the ∠ EAO = CAB, and ACE = ADO

Now (1) the rectangle AB . AC is given, and AB . AD is given (III. xxxvi.); therefore the ratio AB . AC : AB . AD is given; ∴ the ratio AC : AD is given. Again, since the △ˢ ACE, ADO are equiangular; ∴ AC : AE : : AD : AO; alternation, AC : AD : : AE : AO; but the ratio AC : AD is given,



∴ the ratio AE : AO is given; and AO is given, since it is drawn from a fixed point to the centre of a fixed ⊙; ∴ AE is given in magnitude, and it is given in position, because it is drawn making a given ∠ with a given line; hence the point E is given. And because the △ˢ AOD, AEC are equiangular, AO : OD : : AE : EC; but the ratio AO : OD is given; ∴ the ratio AE : EC is given, and AE is given; ∴ EC is given, and the point E has been shown to be given. Hence the locus of C is a ⊙, having E as centre and EC as radius.

15. (1) Let the vertex A remain fixed. Let the locus of B be a right line DB. It is required to find the locus of C.

Sol.—From A let fall a ⊥ AD on DB. Make the ∠ DAG = CAB. Let fall CG ⊥ on AG, and join DG.

Now because the $\angle$ CAB = DAG, the $\angle$ CAG = DAB, and the right $\angle$ CGA = BDA; hence the $\triangle^s$ CAG, DAB are equiangular; $\therefore$ AC : AG : : AB : AD; alternation, AC : AB : : AG : AD; but the ratio AC : AB is given, since the $\triangle$ ABC is given in species; $\therefore$ the ratio AG : AD is given, and AD is given in magnitude, because it is a $\perp$ from a given point on a given line; $\therefore$ AG is given in magnitude, and it is also given in position, since the $\angle$ DAG is equal to a given $\angle$ CAB; $\therefore$ G is a fixed point, and CG is at right $\angle^s$ to a given line at a given point. Hence the locus of C is the line CG.

(2) Let the point B move along a $\odot$; let O be its centre. Join AO, BO, and draw AD, making the $\angle$ DAO = CAB. Draw CD, making the $\angle$ ACD = ABO. Now the $\triangle^s$ ACD, ABO are equiangular; $\therefore$ AC : AD : : AB : AO; alternation, AC : AB : : AD : AO; but the ratio AC : AB is given; $\therefore$ the ratio AD : AO is given, and AO is given; $\therefore$ AD is given. And since it makes the $\angle$ DAO = CAB with a given line AO, $\therefore$ AD is given in position; hence the point D is given. Again, in the $\triangle^s$ AOB, ADC we have AO : OB : : AD : DC; but the ratio AO : OB is given; $\therefore$ the ratio AD : DC is given, and AD is given; $\therefore$ DC is given, and the point D is given. Hence the locus of C is a $\odot$, having D as centre and DC as radius.

16. (1) **Dem.**—Bisect the sides BC, CA, AB in D, E, F. Join AD, BE, CF; let them intersect in O. Produce AD to G, so that DG = OD. Join BG. Draw EH $\parallel$ to AG, and produce BG to meet it in H.

Now since BD = CD, the $\triangle$ BDO = CDO, and the $\triangle$ BDA = CDA; $\therefore$ the $\triangle$ BOA = COA. In like manner, COA = COB, $\therefore$ the $\triangle^s$ AOB, AOC, BOC are equal; $\therefore$ AOB = $\frac{1}{3}$ ABC. And because OG = OA, the $\triangle$ BOG = AOB; hence BOG = $\frac{1}{3}$ ABC. And since the $\triangle^s$ BOG, BEH are similar, BOG : BEH : : OB^2 : BE^2 (XIX.); $\therefore$ BOG : BEH : : 4 : 9; that is, $\frac{1}{3}$ ABC : BEH : : 4 : 9; hence 4 BEH = 3 ABC, $\therefore$ ABC = $\frac{4}{3}$ BEH. Again, it is evident that the sides of the $\triangle$ BEH are equal to the medians of ABC; hence, denoting the medians by α, β, γ, and their half sum by σ, we have (IV. IV., Ex. 12) the $\triangle$ BEH

$$=\sqrt{\sigma \,.\, \sigma-\alpha \,.\, \sigma-\beta \,.\, \sigma-\gamma}.$$

Hence the $\triangle$ ABC is equal to

$$\tfrac{4}{3}\sqrt{\sigma \,.\, \sigma-\alpha \,.\, \sigma-\beta \,.\, \sigma-\gamma}.$$

(2) **Dem.**—Let Δ denote the area of the triangle; then (IV. IV., Ex. 4) $\Delta^2 = s.s - a.s - b.s - c$; $\therefore 16\,\Delta^2 = (a + b + c)\,(b + c - a)\,(c + a - b)\,(a + b - c)$.

Again, denoting the $\perp^s$ by p', p'', p''', we have $ap' = 2\,\Delta$, $bp'' = 2\,\Delta$, and $cp''' = 2\,\Delta$; $\therefore\ (a + b + c) = \frac{2\,\Delta}{p'} + \frac{2\,\Delta}{p''} + \frac{2\,\Delta}{p'''}$ $= 2\,\Delta\left(\frac{1}{p'} + \frac{1}{p''} + \frac{1}{p'''}\right)$; and, substituting, we get

$$16\,\Delta^2 = 2\,\Delta\left\{\frac{1}{p'} + \frac{1}{p''} + \frac{1}{p'''}\right\}2\,\Delta\left\{\frac{1}{p''} + \frac{1}{p'''} - \frac{1}{p'}\right\}$$
$$2\,\Delta\left\{\frac{1}{p'''} + \frac{1}{p'} - \frac{1}{p''}\right\}2\,\Delta\left\{\frac{1}{p'} \quad \frac{1}{p''} - \frac{1}{p'''}\right\};$$

hence

$$\frac{1}{\Delta^2} = \left(\frac{1}{p'} + \frac{1}{p''} + \frac{1}{p'''}\right)\left(\frac{1}{p''} + \frac{1}{p'''} - \frac{1}{p'}\right)$$
$$\left(\frac{1}{p'''} + \frac{1}{p'} - \frac{1}{p''}\right)\left(\frac{1}{p'} + \frac{1}{p''} - \frac{1}{p'''}\right);$$

and hence

$$\Delta = \frac{1}{\sqrt{\left(\frac{1}{p'}+\frac{1}{p''}+\frac{1}{p'''}\right)\left(\frac{1}{p''}+\frac{1}{p'''}-\frac{1}{p'}\right)\left(\frac{1}{p'''}+\frac{1}{p'}-\frac{1}{p''}\right)\left(\frac{1}{p'}+\frac{1}{p''}-\frac{1}{p'''}\right)}}$$

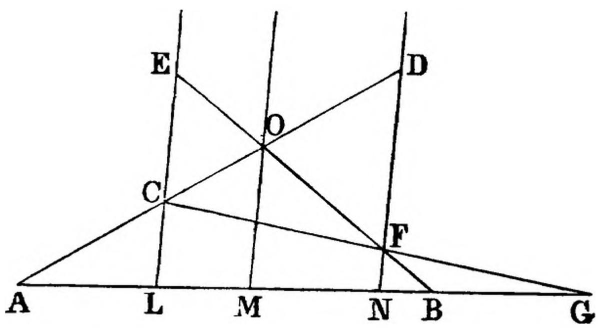
17. Let the $\odot^s$ ABC, DBE touch at B. Draw a common tangent AD. Join, AB, DB, and produce them to meet the $\odot^s$ in E, C. Join DE, AC. DE, AC are the diameters of the $\odot^s$ (III. XIII., Ex. 4).

Now the $\angle$ ADC = AED (III. XXXII.), and the right $\angle$ CAD = ADE; therefore the Δ^s CAD, ADE are equiangular. Hence CA : AD : : AD : DE; that is, AD is a mean proportional between AC and DE.

18. Let CL, OM, FN be the three parallel lines. Take any point O in OM. Join AO, BO, and produce them to meet FN, CL in D, E. Join AB, cutting the $\|^s$ in L, M, N. Join CF, and produce it to meet AB produced in G. It is required to show that G is a given point.

Now in the Δ AOB the line CFG cuts the three sides in C, F, G; hence ("Sequel," Book VI., Prop. IV., Sect. i.), $\frac{AC}{CO}\cdot\frac{OF}{FB}\cdot\frac{BG}{GA} = 1$; but $\frac{AC}{CO} = \frac{AL}{LM}$ (II.), and the ratio $\frac{AL}{LM}$ is

given; $\therefore \frac{AC}{CO}$ is given. In like manner, $\frac{OF}{FB}$ is given; $\therefore \frac{BG}{GA}$ is given. Hence the line AB is divided externally in G in a

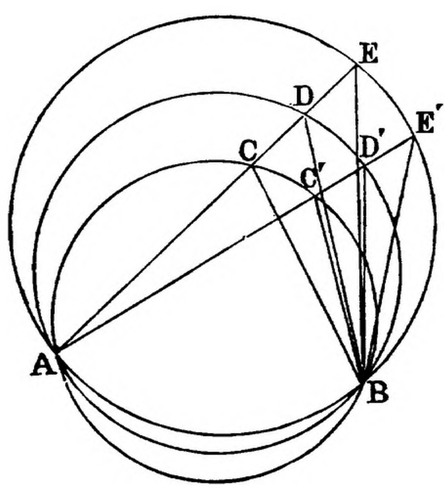


given ratio; $\therefore$ G is a given point. Hence CF passes through a fixed point. Similarly, DE passes through a fixed point.

19. Let a system of ⊙s pass through two fixed points A, B. From A draw any two secants, cutting the ⊙s in C, D, E; C′, D′, E′. It is required to prove that CD : DE : : C′D′ : D′E′.

Dem.—Join BC, BD, BE; BC′, BD′, BE′.

Now the ∠ ACB = AC′B (III. xxi.); $\therefore$ DCB = D′C′B, and CDB = C′D′B; $\therefore$ the △s CDB, C′D′B are equiangular; hence



CD : DB : : C′D′ : D′B. In like manner, the △s DEB, D′E′B are equiangular, and BD : DE : : BD′ : D′E′. Hence *ex aequali* CD : DE : : C′D′ : D′E′.

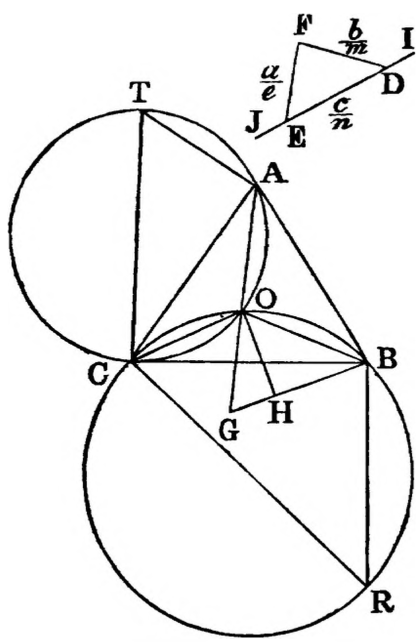
20. Let ABC be a △, the sides being denoted by a, b, c. It is required to find a point O in ABC, such that the diameters of the

⊙ˢ about the △ˢ OAB, OBC, OCA may be in the ratios of three given lines l, m, n.

Sol.—Construct a △ EDF whose sides EF, DF, DE shall be in the ratios $\frac{a}{l}, \frac{b}{m}, \frac{c}{n}$. Produce ED to I, J. On CB describe a segment of a ⊙ containing an ∠ = IDF, and on AC a segment containing an ∠ = JEF. O, where these segments intersect, is the required point.

Dem.—Join OA, OB, OC. Produce AO, and draw BG ∥ to OC. From O let fall a ⊥ OH on BG. Draw CR, CT, the diameters of the ⊙ˢ. Join BR, AT.

Now the sum of the ∠ˢ AOC, GOC is two right ∠ˢ, and the sum of FEJ, FED is two right ∠ˢ; hence GOC = FED; but GOC = OGB (I. xxix.); ∴ OGB = FED. Again, the ∠ˢ COB, GBO equal two right ∠ˢ, and IDF, EDF equal two right ∠ˢ; ∴ GBO = EDF. Hence the △ˢ OBG, DEF are equiangular.



Because the ∠ˢ CTA, COA = two right ∠ˢ (III. xxii.), and COA, COG equal two right ∠ˢ, the ∠ COG = CTA; ∴ OGH = CTA, and OHG = CAT, each being right; ∴ the △ˢ CAT, OGH are equiangular; ∴ $\frac{CT}{CA} = \frac{OG}{OH}$. Again, the

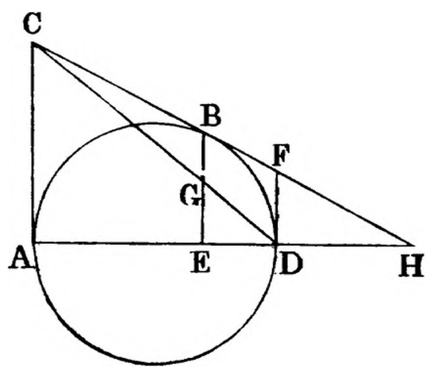
$\angle^s$ COB, OBG equal two right $\angle^s$, and COB, CRB equal two right $\angle^s$; $\therefore$ OBH = CRB, and the right $\angle$ CBR = OHB; $\therefore$ the $\triangle^s$ CBR, OHB are equiangular; $\therefore \frac{\text{CR}}{\text{CB}} = \frac{\text{OB}}{\text{OH}}$. Hence $\frac{\text{CT}}{b} : \frac{\text{CR}}{a}$ :: OG : OB; but OG : OB :: $\frac{a}{l} : \frac{b}{m}$; $\therefore \frac{\text{CT}}{b} : \frac{\text{CR}}{a}$ :: $\frac{a}{l} : \frac{b}{m}$; $\therefore \frac{\text{CT}}{m} = \frac{\text{CR}}{l}$. Hence CR : CT :: $l : m$. In like manner it can be shown that CT is to the diameter of the ⊙ about OAB as m to n.

21. **Sol.**—Describe a ⊙ about ABCD. Join CB, CD, BD. Divide BD at E in a given ratio, and join CE, AC.

Now the points A, C are given, $\therefore$ AC is given in position, and AD is given in position; hence the $\angle$ DAC is given; but (III. XXI.) DAC = DBC; $\therefore$ DBC is a given $\angle$. In like manner, the $\angle$ BDC is given, $\therefore$ the $\angle$ DCB is given; hence the $\triangle$ DBC is given in species; $\therefore$ DB : BC is given, and DB : BE is given (hyp.); $\therefore$ BC : BE is given, and the $\angle$ CBE is given. Hence the $\triangle$ EBC is given in species. Now EBC is a $\triangle$ of given form. One of its vertices, C, is fixed; another, B, moves along a line AB. Hence (Ex. 15) the locus of E is a straight line.

22. **Dem.**—Draw DF ∥ to EB. Produce CF, AD to meet in H.

Now, because DF is ∥ to BG, we have DF : BG :: CF : CB; but DF = BF; $\therefore$ BF : BG :: CF : CB.



Again, since the lines AC, BE, FD are parallel, we have (II., Ex. 1) BF : DE :: CF : AD; and, by similar $\triangle^s$, DE : EG :: AD : AC; hence, *ex aequali*, BF : EG :: CF : AC; but AC

= CB; $\therefore$ BF : EG :: CF : CB. But it has been proved that BF : BG :: CF : CB, therefore BG = EG.

Lemma.—Take any point O within a $\triangle$ ABC. Join OA, OB, OC, and produce AO to meet BC in A′. It is required to prove that the $\triangle$ OBC : ABC :: OA′ : AA′.

Dem.—From A, O let fall $\perp^s$ AD, OE on BC.

Now the $\triangle$ ABC $= \frac{1}{2}$BC.AD, and the $\triangle$ OBC $= \frac{1}{2}$BC.OE; hence ABC : OBC :: AD : OE; but AD : OE :: AA′ : OA; $\therefore$ ABC : OBC :: AA′ : OA′.

23. **Dem.**—The $\triangle^s$ OBC + OCA + OAB = ABC. Divide by ABC, and we have

$$\frac{OBC}{ABC} + \frac{OCA}{ABC} + \frac{OAB}{ABC} = 1; \text{ but } \frac{OBC}{ABC} = \frac{OA'}{AA'} \text{ (\textit{Lemma})};$$

and similarly for the others. Hence

$$\frac{OA'}{AA'} + \frac{OB'}{BB'} + \frac{OC'}{CC'} = 1.$$

24. **Dem.**—AB : BC :: $\triangle$ AOB : BOC (I.), and A′B′ : B′C′ :: $\triangle$ A′OB : $\triangle$ B′OC′; $\therefore$ (Book V., Ex. 5)

$$\frac{AB}{A'B'} : \frac{BC}{B'C'} :: \frac{AOB}{A'OB'} : \frac{BOC}{B'OC'};$$

but (XXIII., Ex. 1),

$$\frac{AOB}{A'OB'} : \frac{BOC}{B'OC'} :: \frac{AO.OB}{A'O.OB'} : \frac{OB.OC}{OB'.OC'}; \therefore \frac{AB}{A'B'} : \frac{BC}{B'C'}$$

$$:: \frac{AO.OB}{A'O.OB'} : \frac{OB.OC}{OB'.OC'}; \therefore \frac{AB}{A'B'} : \frac{BC}{B'C'} :: \frac{AO}{A'O} : \frac{OC}{OC'}.$$

Hence

$$\frac{AB}{A'B'}\cdot\frac{OC}{OC'} = \frac{BC}{B'C'}\cdot\frac{OA}{OA'}.$$

And similarly,

$$\frac{BC}{B'C'}\cdot\frac{OA'}{OA} = \frac{CA}{C'A'}\cdot\frac{OB}{OB'}.$$

25. (1) **Dem.**—Draw the diagonals AC, BD. Bisect them in F, E. Join FE, and produce both ways to meet AD, BC, and DC produced in H, G, I. Now, in the $\triangle$ BDC, the line EI

cuts the three sides in E, G, I. Hence ("Sequel," Book VI. Prop. IV., Sect. i.)

$$\frac{BE}{ED} \cdot \frac{DI}{IC} \cdot \frac{CG}{GB} = 1;$$

but

$$\frac{BE}{ED} = 1, \therefore \frac{DI}{IC} \cdot \frac{CG}{GB} = 1; \therefore \frac{DI}{IC} = \frac{GB}{CG}.$$

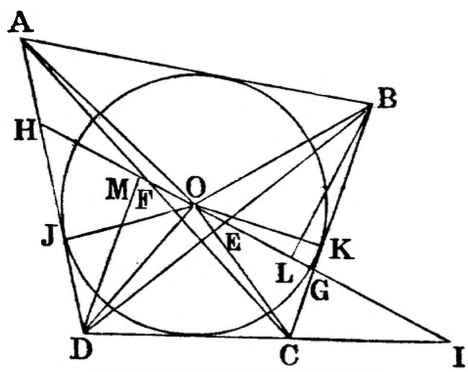
In like manner, from the △ ADC, we get

$$\frac{DI}{IC} = \frac{HD}{AH}.$$

Hence

$$\frac{GB}{CG} = \frac{HD}{AH}.$$

(2) **Dem.**—Join O, the centre, to A, B, C, D. Join O to J, K,



where AD, BC touch the ⊙. Now, since OK = OJ, we have (I.) the △ OBC : OAD :: BC : AD. Let fall ⊥ˢ BL, DM on OG, OH; then (I. XXVI.) the △ˢ BEL, DEM are equal; ∴ BL = DM, and ∴ the △ OBG : OHD :: OG : OH. In like manner OCG : OHA :: OG : OH. Adding, we have OBC : OAD :: OG : OH; but it was shown that OBC : OAD :: BC : AD. Hence BC : AD :: OG : OH.

(3) **Dem.**—Consider the △ ECI. It is intersected by AB; hence ("Sequel," Book VI., Prop. IV., Sect. i.)

$$\frac{EG}{GI} \cdot \frac{IB}{BC} \cdot \frac{CA}{AE} = 1; \text{ but } \frac{CA}{AE} = 2; \therefore \frac{EG}{GI} \cdot \frac{IB}{BC} = \tfrac{1}{2}.$$

Again, consider the △ AEK; it is intersected by CD;

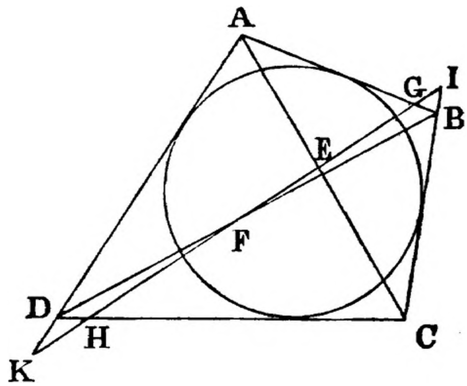
$$\therefore \frac{EH}{HK} \cdot \frac{KD}{DA} \cdot \frac{AC}{CE} = 1;$$

and, as before,

$$\frac{EH}{HK} \cdot \frac{KD}{DA} = \frac{1}{2}; \therefore \frac{EG}{GI} \cdot \frac{IB}{BC} = \frac{EH}{HK} \cdot \frac{KD}{DA}.$$

Now AD, BC are opposite sides, and they are cut by EF in K, I; hence (1) they are cut proportionally;

$$\therefore \frac{CB}{IB} = \frac{AD}{DK}, \text{ and } \therefore \frac{EG}{GI} = \frac{EH}{HK};$$



that is, EG : GI : : EH : HK; and the first is to the sum of the first and second as the third is to the sum of the third and fourth. Hence EG : EI : : EH : EK.

26. It is required to prove that AD . DB : AC . CB : : AD^2 : AC^2.

Dem.—AD . DB, AC . CB, are rectangular figures; and since AD : DB : : AC : CB (III.), these figures are similar; hence (XIX.) AD . DB : AC . CB : : AD^2 : AC^2. In like manner AC . CB : AD′ . D′B : : AC^2 : AD'^2.

(1) **Dem.**—If AD . DB, AC . CB, and AD′ . D′B, are in A. P., the difference between AD . DB and AC . CB is equal to the difference between AC . CB and AD′ . D′B; but AC . CB − AD . DB = CD^2 (XVII., Ex. 1), and AD′ . D′B − AC . CB = CD'^2; ∴ CD^2 = CD'^2, ∴ CD = CD′, ∴ the ∠ CDD′ = CD′D; but the ∠ DCD′ is right; ∴ each of the ∠s CDD′, CD′D is half a right ∠; hence the ∠ CDA is a right ∠ and a-half. Now the ∠ CDA = CBD + BCD, and CDB = CAD + ACD; hence CDA − CDB = CBD − CAD: but the difference between CDA and CDB is a right ∠. Hence the difference between CBD and CAD is a right ∠.

(2) **Dem.**—If the three rectangles be in G. P., the squares of the lines DB, BC, BD′ are in G. P.; ∴ DB, BC, BD′ are in G. P., ∴ BC is a mean proportional between DB and BD′; but the ⊥ is a mean proportional between the segments of the hypotenuse (VIII., Cor. 1). Hence BC is a ⊥, and hence the ∠ ABC is right.

(3) **Dem.**—If the rectangles AD.DB, AC.CB, AD′.D′B are in H. P., the 1st : 3rd :: difference between 1st and 2nd : difference between 2nd and 3rd; but difference between 1st and 2nd = CD^2 (XVII., Ex. 1) and difference between 2nd and 3rd = CD'^2, ∴ AD.DB : AD′.D′B :: $CD^2 : CD'^2$; but, by similar figures, AD.DB : AD′.D′B :: $DB^2 : D'B^2$; hence $CD^2 : CD'^2 :: DB^2 : D'B^2$; ∴ CD : CD′ :: DB : D′B, and ∴ (III.) the ∠ DCD′ is bisected, ∴ the ∠ DCB is half a right ∠; but the ∠ ACD = DCB; ∴ the ∠ ACB is right. Hence the sum of the ∠s CAB, CBA is a right ∠.

28. **Dem.**—Denote the radii of the ⊙s by ρ, ρ'; then (VI. IV.) DC : D′C :: $\rho : \rho'$, and A′C : BC :: $\rho : \rho'$, ∴ DC : D′C :: A′C : BC; ∴ DD′ : D′C :: A′B : BC. (V. XVII.) In like manner DD′ : D′C :: AB′ : B′C, ∴ $DD'^2 : D'C^2$ :: A′B.AB′ : BC.B′C; but $D'C^2$ = BC.B′C (III. XXXVI.). Hence DD'^2 = A′B.AB′.

29. **Dem.**—Because A′O is ∥ to BO″, AO″ : OO″ :: AB : A′B (II.); that is, $R : (R - \rho) :: AB : A'B$. Similarly, $R : (R - \rho') :: AB : AB'$; ∴ $R^2 : (R - \rho)(R - \rho') :: AB^2 : A'B.AB'$; but $A'B.AB' = DD'^2$ (III. XXXVI.). Hence $R^2 : (R - \rho)(R - \rho') :: AB^2 : DD'^2$.

30. **Dem.**—Let A, B, C, D be the points in which the four ⊙s touch the fifth. Join AB, BC, CD, DA, AC, BD, and denote the radii of the four ⊙s by $\rho_1, \rho_2, \rho_3, \rho_4$, and the radius of the fifth by R; then putting $\overline{12}^2$ for DD'^2, we have, from Ex. 29, $AB^2 : \overline{12}^2 :: R^2 : (R - \rho_1)(R - \rho_2)$; hence

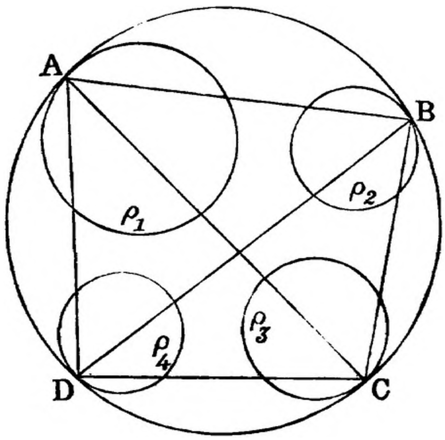
$$AB^2 = \frac{\overline{12}^2 . R^2}{(R - \rho_1)(R - \rho_2)}; \quad \therefore \ AB = \frac{\overline{12} . R}{\sqrt{(R - \rho_1)(R - \rho_2)}}.$$

Similarly,

$$CD = \frac{\overline{34} . R}{\sqrt{(R - \rho_3)(R - \rho_4)}}, \quad AD = \frac{\overline{14} . R}{\sqrt{(R - \rho_1)(R - \rho_4)}},$$

and
$$BC = \frac{\overline{23} . R}{\sqrt{(R - \rho_2)(R - \rho_3)}}.$$

Now, by Ptolemy's theorem (XVII., Ex. 13) AB.CD + BC.AD = AC . BD. Therefore



$$\frac{\overline{12} . \overline{34} . R^2}{\sqrt{(R - \rho_1)(R - \rho_2)(R - \rho_3)(R - \rho_4)}} + \frac{\overline{23} . \overline{14} . R^2}{\sqrt{(R - \rho_2)(R - \rho_3)(R - \rho_1)(R - \rho_4)}}$$

$$= \frac{\overline{13} . \overline{24} . R^2}{\sqrt{(R - \rho_1)(R - \rho_3)(R - \rho_2)(R - \rho_4)}};$$

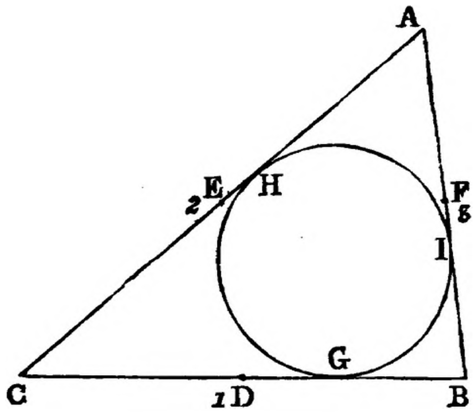
and hence

$$\overline{12} . \overline{34} + \overline{23} . \overline{14} = \overline{13} . \overline{24}.$$

31. **Dem.**—Bisect the sides of the △ ABC in the points D, E, F. Inscribe a ⊙ in ABC, touching the sides in G, H, I. Let the sides opposite the angular points be denoted by a, b, c.

Now if we consider the points D, E, F as infinitely small ⊙s, DE, EF, FG are common tangents to the ⊙s 1, 2; 2, 3; 3, 1; hence we have $\overline{12}$ = DE = $\frac{1}{2}$ AB = $\frac{1}{2} c$. Similarly, $\overline{23} = \frac{1}{2} a$, $\overline{31} = \frac{1}{2} b$.

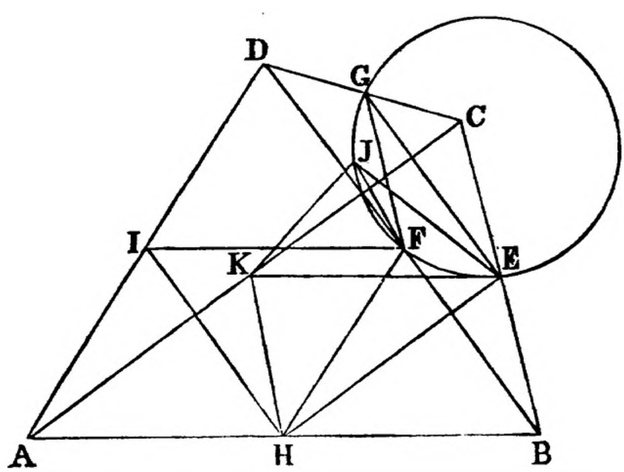
Let the inscribed ⊙ be denoted by 4. Now BD = $\frac{1}{2}$ BC = $\frac{1}{2} a$, and BG = $(s - b)$ (IV. IV., Ex. 2); ∴ DG = $\frac{1}{2} a - (s - b)$ = $\frac{1}{2}(b - c)$; that is, $\overline{14} = \frac{1}{2}(b - c)$. In like manner, $\overline{24} = \frac{1}{2}(c - a)$, and $\overline{34} = \frac{1}{2}(a - b)$. Now if we substitute these values in the conditions of the last question, we find that it is fulfilled. Hence the



⊙ through the middle points of the sides of the △ touches the inscribed circle. Similarly, it touches the escribed circle.

32. Let A, B, C, D be the four points; join them, and join AC, BD. Bisect BC, BD, CD in E, F, G. Bisect AB, AD in H, I. Describe a ⊙ through the points E, F, G, and another ⊙ through H, I, F; let them intersect in J. It is required to prove that the ⊙s through the middle points of the △s ABC, ADC will also pass through J.

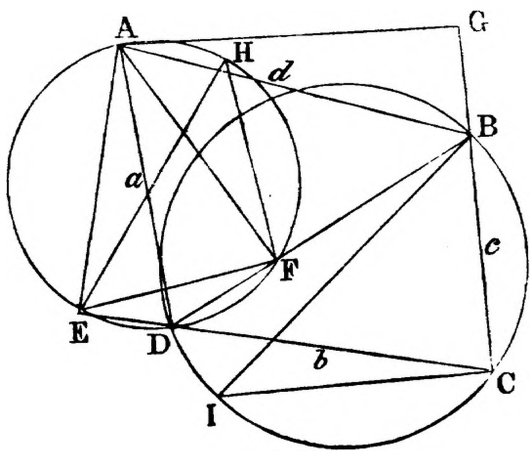
Dem.—Bisect AC in K. Join KE, KH, EH, GE, EJ, JF, H, HI, IF, JK.



Now because CB, CD are bisected in E, G, EG is ∥ to BD. Similarly, GF is ∥ to BC; hence BEGF is a parallelogram; ∴ the ∠ FGE = FBE; but FGE = FJE (III. xxi.), ∴ FJE = FBE. Again, as before, HIFB is a parallelogram, ∴ the ∠ HIF = HBF; but HIF = HJF (III. xxi.);

∴ HJF = HBF; ∴ the whole ∠ HJE = HBE; but HBE = HKE, since HKEB is evidently a parallelogram; ∴ HJE = HKE; hence the four points H, K, J, E are concyclic, and the ⊙ through H, K, E will pass through J. Similarly, the ⊙ through K, I, G will pass through J. Hence the four nine-points ⊙s have a common point.

33. **Dem.**—From A let fall ⊥s AE, AF, AG on CD, DB, CB. Now because the ∠s AED, AFD are right, AEDF is a cyclic quadrilateral, and AD is the diameter of its circumcircle. Draw another diameter EH. Join EF, FH. About the Δ BDC describe a ⊙. Draw its diameter BI, and join IC. Now (III. xxii.) the sum of the ∠s EHF, EDF is two right ∠s,



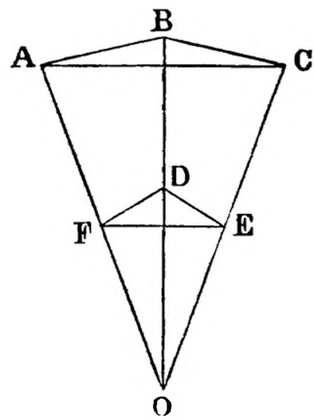
and the sum of EDB, CDB is two right ∠s; hence the ∠ EHF = CDB; but (III. xxi.) CDB = CIB; hence EHF = CIB, and the right ∠ EFH = ICB; ∴ the Δs EFH, ICB are equiangular; hence EH : EF : : IB : BC; ∴ EH . BC = EF . IB; that is, ac = EF . IB. Similarly, bd = FG.IB, and DD′ = EG.IB. Hence EF, FG, EG are proportional to ac, bd, DD′.

34. OEDF is a four-sided figure; OD, EF its diagonals. If OF . DE + OE . DF = OD . EF, it is required to prove that OEDF is a cyclic quadrilateral.

Dem.—Produce OD, OE, OF to B, C, A until each of the rectangles OD . OB, OE . OC, OF . OA is equal to the square of a given line, say R^2. Join AB, BC, AC.

Now OD . OB = OE . OC; ∴ OB : OC : : OE : OD, and the ∠ BOC is common to the two Δs OBC, OED; hence (vi.) they are equiangular, and BC : OB : : ED : OE; alternation, BC : ED : :

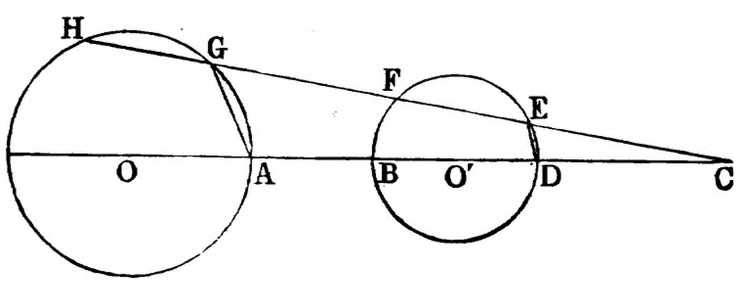
OB : OE ; ∴ BC : ED : : OB.OD : OE . OD ; that is, BC : ED : : R^2 : OE . OD ; hence $\frac{ED}{OE.OD} = \frac{BC}{R^2}$. In like manner, $\frac{DF}{OF.OD} = \frac{AB}{R^2}$, and $\frac{EF}{OE.OF} = \frac{AC}{R^2}$. Now (hyp.) ED . OF + DF . OE = OD . EF ; ∴ $\frac{ED}{OE.OD} + \frac{DF}{OD.OF} = \frac{EF}{OE.OF}$; that is, $\frac{BC}{R^2} + \frac{AB}{R^2} = \frac{AC}{R^2}$; ∴ BC + AB = AC ; but this could not be true unless AB



and BC are in one straight line ; ∴ ABC is a straight line ; ∴ the sum of the ∠s ABO, CBO is two right ∠s ; but ABO = DFO, and CBO = DEO ; ∴ DFO + DEO = to two right ∠s. And hence OEDF is a cyclic quadrilateral.

Lemma.—If C be the external centre of similitude of two ⊙s ; CH any line passing through C, and cutting both ⊙s in the points E, F ; G, H ; it is required to prove that CG . FC = AC . BC.

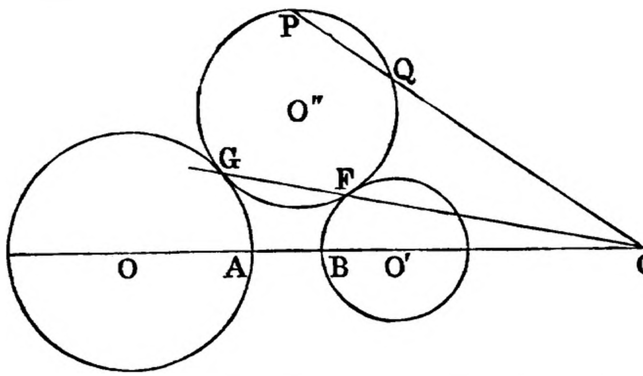
Dem—Join AG, DE.



Now AC : DC : : GC : EC ; ∴ AC.BC : BC.DC : : GC.FC : FC.FC ; but BC . DC = EC.FC. Hence AC . BC = GC . FC.

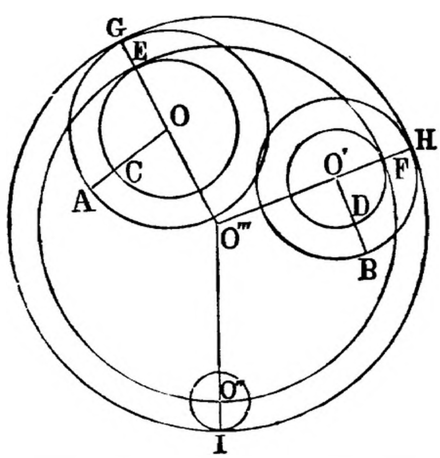
35. (1) Let O, O′ be the centres of the given ⊙s, and P the point.

Sol.—Join OO′, and produce. Let C be the external centre of similitude. Join PC, and find the point Q, so that PC . QC = AC . BC. Describe a ⊙ passing through P, Q, and touching the ⊙ whose centre is O in G (III. xxxvii., Ex. 1). This is the required circle.



Dem.—Join GC, cutting the circle whose centre is O′ in F. Now (const.) PC . QC = AC . BC, and (*Lemma*) AC . BC = GC . FC; ∴ PC . QC = GC . FC. Hence the ⊙ through the points P, Q, G passes through F, and touches the ⊙ whose centre is O′.

(2) **Sol.**—Let O, O′, O″ be the centres of the given ⊙s. Draw any two radii OA, O′B. Cut off AC, BD, each equal to the radius of O″. With O as centre and OC as radius, describe a ⊙. With O′ as centre and O′D as radius, describe a ⊙. Now (1) describe



a ⊙ touching those two in E, F, and passing through the point O″. Let O‴ be its centre. Join O‴O, O‴O′, O‴O″, and produce them to meet the circumference of the given ⊙s in the points G, H, I. The ⊙ through G, H, I will be the required circle.

Dem.—Because OG = OA and OE = OC, EG = AC; but AC = O″I; ∴ EG = O″I, and O‴E = O‴O″; hence O‴G = O‴I. In like maner, O‴H = O‴I. Hence the ⊙ described with O‴ as centre, and O‴G as radius, will pass through H, I, and touch the given ⊙s in the points G, H, I.

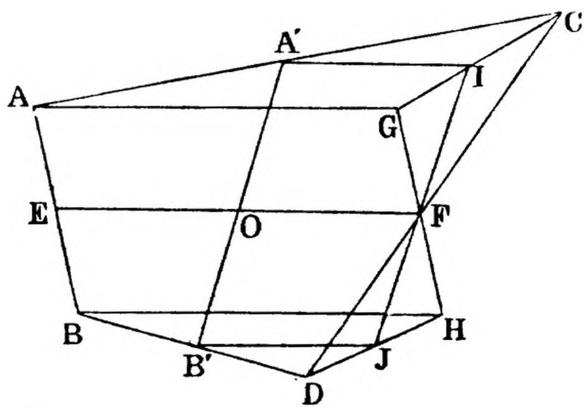
36. Let O, O′ be the centres of the fixed ⊙s, and C their centre of similitude; and let any variable ⊙ O″ touch O, O′ in G, F. From C draw CD a tangent to O″. It is required to prove that CD is of constant length. (See Diagram to Ex. 35 (1)).

Dem.—Join GF, and produce it to pass through C.

Now CD^2 = GC.CF (III. xxxvi.), and GC.CF = AC.CB (*Lemma* to 35); hence CD^2 = AC.CB; but AC.CB is constant, since A, C, B are fixed points. Hence CD is constant.

37. **Dem.**—Draw DD′ a common tangent to the two fixed ⊙s. Join AD, BD′, and produce them; they must meet on the circumference of O″. For, if not, let AD meet the circumference of O″ in P, and BD′ meet it in Q. Join O″O, O″O′, and produce them; O″O, O″O′ must pass through A, B (III. xi.). Join OD, O′D′, O″P, O″Q. Now the ∠ O″AP = O″PA, and OAD = ODA; ∴ ODA = O″PA; hence OD is parallel to O″P. Now the ∠ ODD′ is right (III. xviii.); hence O″P is ⊥ to DD′. Similarly, O″Q is ⊥ to DD′, which is impossible, unless Q coincide with P. Hence BD′ must pass through P.

38. Join A′B′. Take a fixed point C in AC, and in BD find a point D, so that as AA′ : AC : : BB′ : BD. Join AB, and divide it in E in a given ratio. Join CD, and divide it in F in the same ratio. Join EF, cutting A′B′ in O. It is required to prove that A′O : OB′ : : AE : EB.



Dem.—Through F draw GH parallel to AB, and draw AG, BH, each parallel to EF. Join CG, DH. Draw A′I parallel to AG, and B′J parallel to BH. Join IF, JF.

Now, by construction, AA′ : AC :: BB′ : BD ; ∴ AC : A′C :: BD : B′D. And hence, by similar triangles, GC : IC :: DH : DJ ; but GC : CF :: DH : DF. Hence IC : CF :: DJ : DF, and the contained angles ICF, JDF are equal, ∴ the triangles ICF, JDF are equiangular, ∴ the $\angle$ IFC = JFD ; ∴ IF, FJ are in the same straight line.

Again, from similar $\triangle^s$, AG : A′I :: AC : A′C, and BH : B′J :: DB′ : DB; hence AG : A′I :: BH : B′J ; but AG = BH ; ∴ A′I = B′J ; hence IJ is parallel to A′B′ ; ∴ AO′ : OB′ :: IF : FJ ; that is, :: CF : FD, or :: AE : EB. Hence the locus of the point in which A′B′ is divided in the ratio of AE : EB is the right line EF.

39. **Dem.**—It was proved in the last Exercise that A′O : OB :: AE : EB. In like manner, EO : OF :: AA′ : A′C. Now putting G, H for A′, B′, we have GO : OH :: AE : EB, and EO : OF :: AG : GC.

Lemma.—If a given line AC be divided in B, so that $AB . BC^4$ is a maximum ; it is required to prove that BC = 4 AB.

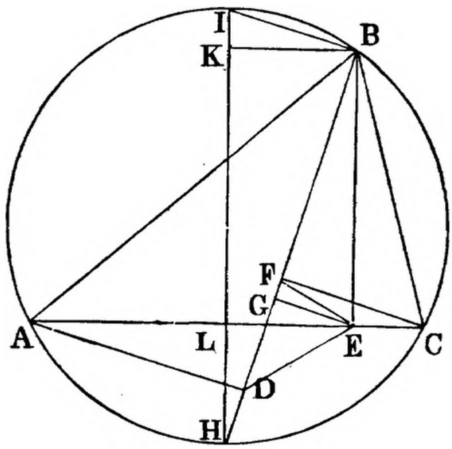
Dem.—Divide BC into four equal parts in E, F, G; then each of the parts BE, EF, FG, GC is equal to $\frac{BC}{4}$; hence $BE . EF . FG . GC = \frac{BC^4}{256}$. Multiply each by AB, and we get

$AB . BE . EF . FG . GC = \frac{AB . BC^4}{256}$; but (hyp.) $AB . BC^4$ is a maximum ; ∴ AB . BE . EF . FG . GC is a maximum ; ∴ AB, BE, EF, FG, GC are all equal ("Sequel," Book II., Prop. XII., Cor.). Hence BC = 4 AB.

Similarly, if it be required to divide AC in B, so that $AB . BC^n$ may be a maximum, BC = nAB.

40. *Analysis.*—Let ABC be the required $\triangle$. Bisect the vertical $\angle$ ABC by BH. From A, C let fall $\perp^s$ AD, CF on BH, and from B let fall a $\perp$ BE on AC. Join DE, EF. Draw HI, the diameter. Join BI. Draw BK $\parallel$ to AC, and let fall a $\perp$ EG on HB.

Now the $\angle$ ADB = AEB, each being right; hence the four points A, D, E, B are concyclic; $\therefore$ the $\angle$ EDF = BAC. Again, because each of the $\angle^s$ BEC, BFC is right, BFEC is a cyclic quadrilateral; $\therefore$ the sum of the $\angle^s$ BFE, BCE is two right $\angle^s$, and the sum of BFE, DFE is two right $\angle^s$; $\therefore$ the $\angle$ DFE = BCA; $\therefore$ the $\triangle^s$ ABC, DEF are equiangular. And since

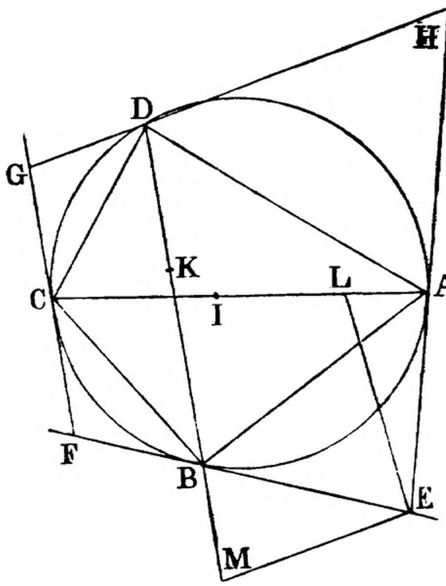


their $\perp^s$ are BE, EG, ABC : DEF :: $BE^2 : EG^2$; but $BE^2 : EG^2 :: HI^2 : IB^2$, or HI . IK; $\therefore$ $BE^2 : EG^2$ :: HI : IK; $\therefore$ ABC : DEF :: HI : IK; $\therefore$ ABC . IK = DEF . HI. Now DEF is a maximum (hyp.), and HI is a given line, because it is the diameter of the $\odot$; $\therefore$ ABC . IK is a maximum. Now ABC = $\frac{1}{2}$ base . perpendicular = AL . BE, or AL . KL; $\therefore$ AL . KL . IK is a maximum. Now whatever AL is, the rectangle KL . IK is a maximum when IL is bisected in K, and then KL . KI = $\frac{1}{4}$ IL^2; $\therefore$ AL . $\dfrac{IL^2}{4}$ is a maximum; $\therefore$ AL . IL^2 is a maximum; $\therefore$ $AL^2 . IL^4$ is a maximum; but AL^2 = HL . LI; $\therefore$ HL . IL^5 is a maximum. And $\therefore$ (*Lemma*) IL = 5 HL. Hence the method of construction is evident.

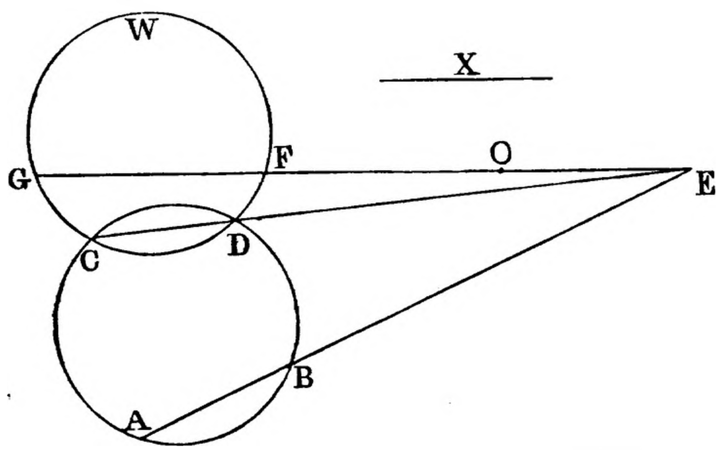
41. Let AC, BD, the diagonals of the inscribed quadrilateral, intersect in O. At the points A, B, C, D draw tangents to the $\odot$. Let them meet in E, F, G, H; then EFGH is a circumscribed quadrilateral. It is required to prove that its diagonals EG, FH must pass through O.

Dem.—If possible let EG not pass through O; but cut AC, BD in I, K. Produce AE, CF to meet in J (not represented in the diagram). Through E draw EL $\parallel$ to GJ, and EM $\parallel$ to GD.

Produce DB to meet EM. Now because JA = JC being tangents, the $\angle$ JCA = JAC; but ELA = JCA (I. xxix.); $\therefore$ EAL = ELA; and $\therefore$ EA = EL. In like manner EB = EM; but EA = EB;



$\therefore$ EL = EM. Now since the $\triangle$s GCI, ELI are equiangular, GC : GI : : EL : EI; alternation, GC : EL : : GI : EI; but GC = GD, and EL = EM; $\therefore$ GD : EM : : GI : EI; and because the $\triangle$s GKD, MKE are equiangular, GD : EM : : GK : EK; $\therefore$ GI : EI : : GK : EK, which is impossible unless the points I, K coincide. Hence GE must pass through O. In like manner FH must pass through O.



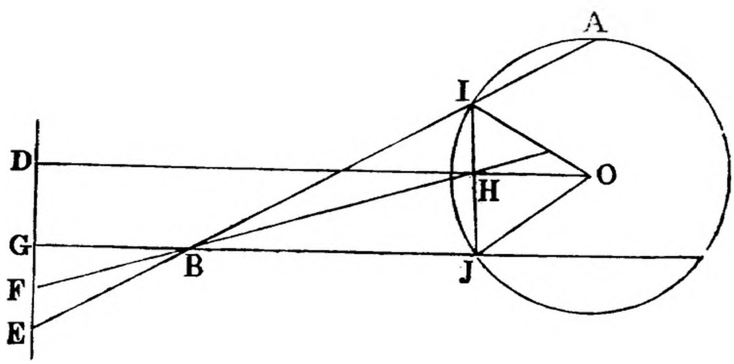
42. (1) **Sol.**—Let A, B be the given points, W the given $\odot$,

and X the given line. Through A, B describe any ⊙ cutting W in C, D. Join AB, CD, and produce them to meet in E. Through E draw EFG parallel to X, and cutting W in F, G. The ⊙ through A, B, F, G is the one required.

Dem.—AE . EB = CE . ED, and CE . ED = GE . EF; ∴ AE . EB = GE . EF. Hence the four points A, B, F, G are concyclic, and the common chord FG is ∥ to X.

(2) **Sol.**—Let O be the given point. Make the same construction as before; and instead of drawing EFG ∥ to X, join EO, and produce it to cut W in F, G. Then, as in (1), EFG is a common chord, and it passes through O, the given point.

43. **Sol.**—Let O be the centre of the ⊙, ABC the ∠, and DE the given line. Produce AB, CB to meet DE in E, G. Bisect GE in F. Join FB. From O let fall a ⊥ OD on DE, and meeting FB produced in H. Through H draw IJ ∥ to DE,



meeting AB, CB in I, J. Join OI, OJ. Now because the lines GJ, FH, EI pass through B, and are cut by the ∥ˢ GE, IJ, GF : FE : : IH : HJ; but GF = FE; ∴ IH = HJ; and since IJ is ∥ to DE, and OD meets them, the ∠ OHJ = ODE; ∴ OHJ is a right ∠; ∴ OHI is right, and ∴ (I. iv.) OJ = OI; and the ⊙, with O as centre, and OJ as radius, will pass through I, and its chord IJ is parallel to the given line DE.

44. Let ABCDE be a polygon of an odd number of sides. Take any point O within it. Join AO, BO, CO, DO, EO, and produce them to meet the opposite sides in A′, B′, C′, D′, E′. It is required to prove that the product of AD′, BE′, CA′, DB′, EC′ is equal to the product of A′D, B′E, C′A, D′B, E′C.

Dem.—Join AC, AD. Now the △ AOD : A′OD : : AO : A′O (i.), and AOC : A′OC : : AO : A′O; ∴ AOD : A′OD : : AOC

: A′OC; alternation, AOD : AOC : : A′OD : A′OC; but A′OD : A′OC : : A′D : A′C. Hence

$$\frac{A'D}{A'C} = \frac{AOD}{AOC}.$$

In like manner, by joining BE, BD; CE, CA; DB, DA; EC, EB, we get

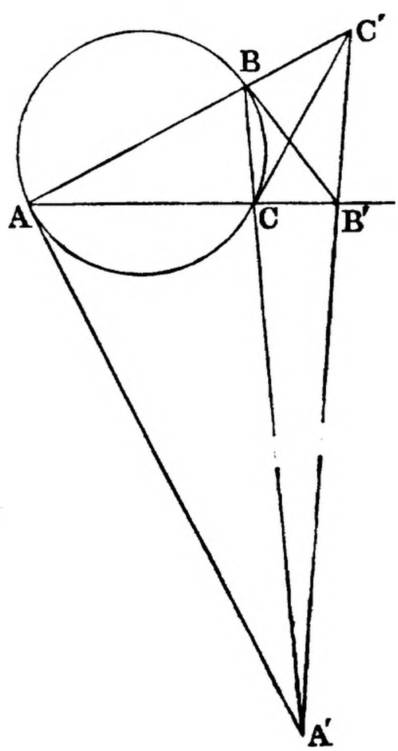
$$\frac{EB'}{BD} = \frac{EOB}{BOD}; \quad \frac{AC'}{C'E} = \frac{AOC}{EOC}; \quad \frac{BD'}{D'A} = \frac{BOD}{DOA}; \quad \frac{CE'}{E'B} = \frac{COE}{BOE}.$$

Now, multiplying these together, we find that the numerators of the second terms are equal to the denominators. Hence the product of the numerators of the first terms is equal to the product of the denominators; that is, A′D . B′E . C′A . D′B . E′C = A′C . B′D . C′E . D′A . E′B.

45. Let ABC be the △, and let the sides touch the ⊙ in the points A′, B′, C′.

Dem.—Join AA′, BB′, CC′. Now AB′ = AC′, BA′ = BC′, CA′ = CB′. Hence AC′. CB′. BA′ = A′C . C′B . B′A; and hence (Ex. 4) the lines AA′, BB′, CC′ are concurrent.

46. From A, B, C draw tangents AA′, BB′, CC′; and produce

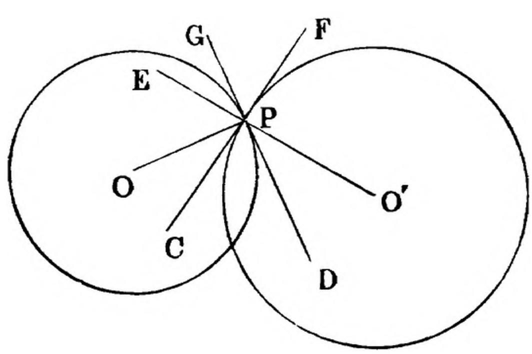


the sides BC, AC, AB to meet them in A′, B′, C′. It is required to show that the points A′, B′, C′ are collinear.

Dem.—The ∠ B′BC = BAB′ (III. xxxii.), and the ∠ BB′C is common, ∴ the ∠s AB′B, BB′C are equiangular, ∴ AB′ : AB : : BB′ : BC; alternation, AB′ : BB′ : : AB : BC, ∴ $AB'^2 : BB'^2 :: AB^2 : BC^2$; but $BB'^2 = AB' . B'C$ (III. xxxvi.), ∴ $AB'^2 : AB . B'C :: AB^2 : BC^2$, ∴ $AB' : B'C :: AB^2 : BC^2$. Hence, denoting the sides of the Δ ABC by a, b, c, we have $AB' : B'C :: c^2 : a^2$. Interchange, and we get $BC' : C'A :: a^2 : {}^2$, and $CA' : A'B :: b^2 : c^2$. Multiply these together, and we have $AB' . BC' . CA' : B'C . C'A . A'B :: c^2 a^2 b^2 : a^2 b^2 c^2$; ∴ $AB' . BC' . CA' = B'C . C'A . A'B$; and hence (Ex. 5) the points A′, B′, C′ are collinear.

47. **Dem.**—Produce the sides, and draw AA′, BB′, CC′, bisecting the external ∠s. Now (iii., Ex. 1 AB′ : B′C : : AB : BC. Interchange, and we have BC′ : C′A : : BC : CA. Interchange again, and CA′ : A′B : : CA : AB. Now, multiply together, and AB′ . BC′ . CA′ : B′C . C′A . A′B : : AB . BC . CA : BC . CA . AB; but the third term is equal to the fourth, ∴ the first is equal to the second; that is, AB′ . BC′ . CA′ = B′C . C′A . A′B; and hence (Ex. 5) the points A′, B′, C′ are collinear.

Lemma.—Let two ⊙s, whose centres are O, O′, cut in P. Join

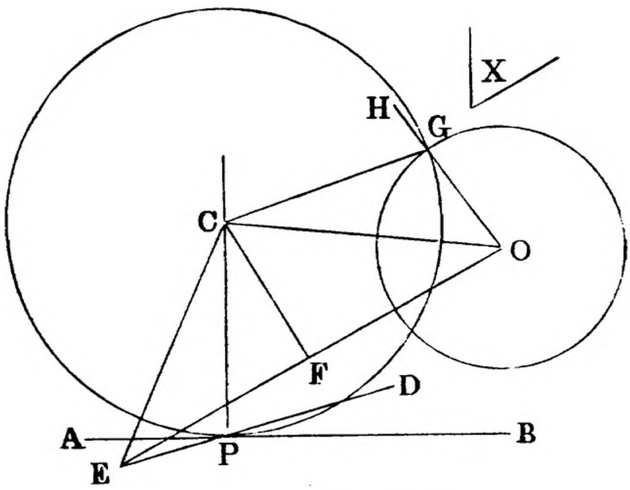


OP, O′P. Produce O′P to E. Draw CP, DP tangents to the ⊙s. It is required to show that the ∠ EPO = CPD.

Dem.—Produce CP, DP to F and G. Now the ∠ O′PF is right (III. xviii.); hence (I. xv.) CPE is right, and OPD is right; ∴ CPE = OPD. Reject OPC, and EPO = CPD.

48. Let AB be a given line, P a given point, O the centre of the given ⊙, and X a given ∠. It is required to describe a ⊙, touching AB in P, and cutting O at an ∠ equal to X.

Sol.—Erect PC ⊥ to AB. Draw DP, making the ∠ CPD = X. Produce DP to E, cut off EP equal to the radius of O. Join EO.



Bisect it in F. Erect FC ⊥ to EO, meeting PC in C. With C as centre, and CP as radius, describe a ⊙, cutting O in G. This is the required circle.

Dem.—Join EC, CO, CG, OG. Now because EF = OF, and FC common, and the ∠ EFC = OFC; ∴ (I. IV.) EC = OC, and CP = CG, being radii, and EP = OG (const.), ∴ the ∠ EPC = OGC; but DPC and EPC are supplements; and HGC, OGC are supplements, ∴ HGC = DPC; but DPC = X, and HGC is equal to the ∠ between the ⊙s (*Lemma*). Hence the ∠ between the ⊙s is equal to the given ∠, and the ⊙ PG touches AB in P.

49. See "Sequel," Book IV. Prop. III., Cor. 2.

50. See "Sequel," Book I. Prop. XVII.

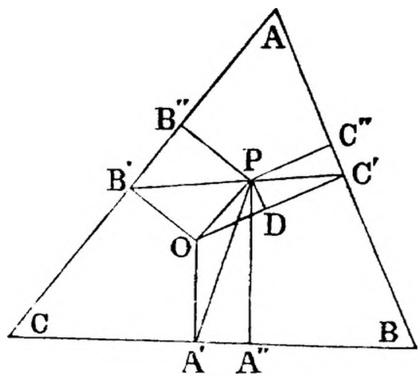
51. See "Sequel," Book II. Prop. X.

52. Let O be the centre of mean position of the feet of ⊥s from it on the sides. From O let fall ⊥s OA′, OB′, OC′ on the sides. Take any other point P within the △, and let fall ⊥s PA″, PB″, PC″ It is required to show that $OA'^2 + OB'^2 + OC'^2$ is less than $PA''^2 + PB''^2 + PC''^2$.

Dem.—Join OP, PA′, PB′, PC′. Now, because O is the centre of mean position of A′, B′, C′, we have (Ex. 51) $A'P^2 + B'P^2$

$+ C'P^2 = OA'^2 + OB'^2 + OC'^2 + 3\,OP^2$; but $A'P^2 = A'A''^2 + A''P^2$, $B'P^2 = B'B''^2 + B''P^2$, and $C'P^2 = C'C''^2 + C''P^2$; $\therefore$ $A'A''^2 + B'B''^2 + C'C''^2 + A''P^2 + B''P^2 + C''P^2 = OA'^2 + OB'^2 + OC'^2 + 3\,OP^2$.

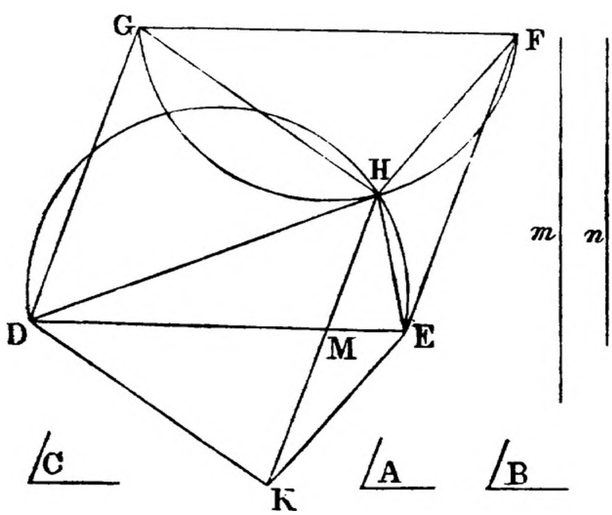
From P let fall a $\perp$ PD on OC′, then OP^2 is greater than PD^2;



that is, greater than $C'C''^2$. In like manner it is greater than $A'A''^2$, and greater than $B'B''^2$, $\therefore$ $3\,OP^2$ is greater than $A'A''^2 + B'B''^2 + C'C''^2$; and hence $A''P^2 + B''P^2 + C''P^2$ is greater than $OA'^2 + OB'^2 + OC'^2$.

53. (1) Let A, B be the opposite $\angle$ s; *m*, *n* the diagonals, and C the angle between the diagonals.

Sol.—Construct a parallelogram DEFG, having two adjacent



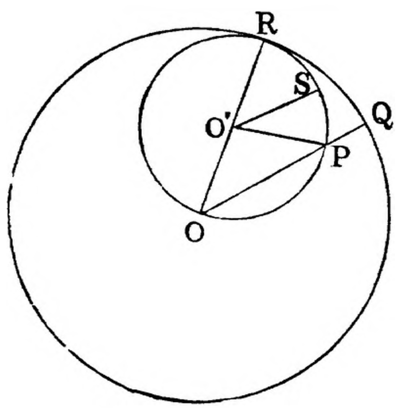
sides DE, DG respectively equal to *m* and *n*, and their included

∠ = to C. On DE describe a segment of a ⊙ containing an ∠ equal to A; and on FG describe a segment containing an ∠ equal to B; let them intersect in H. Join HD, HE, HF, HG. Through H draw HK ∥ and = to EF. Join DK, EK. DHEK is the required quadrilateral.

Dem.—The ∠ DHE = A, and EF = HK (I. xxxiv.); but EF = GD; ∴ HK = GD, and it is ∥ to it; ∴ HKDG is a parallelogram; ∴ HG is ∥ to DK, and HF is ∥ to EK; hence the ∠ GHF = DKE; but GHF = B, ∴ DKE = B; and (I. xxix.) the ∠ HME = GDE; but GDE = C, ∴ HME = C.

54. Let a ⊙, whose centre is O′, roll inside another ⊙, O, whose diameter is twice that of O′. Take a fixed point P in the circumference of O′. It is required to find its locus.

Sol.—Let R be the point of contact. Join OP, OR, O′P, and

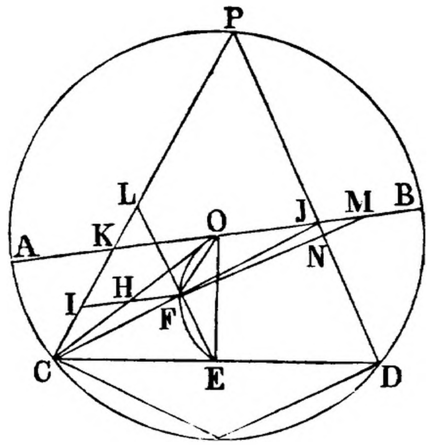


produce OP to meet the circumference in Q, and bisect the ∠ RO′P by O′S.

Now the ∠ RO′P = 2 ROP (III. xx.); ∴ the ∠ RO′S = ROQ, and the arc RS : RQ : : O′R : OR; but OR = 2 O′R, ∴ RQ = 2 RS, ∴ RP = RQ. Now, since the arc RP = RQ, the point P must have coincided with Q. Hence the line OQ is the locus of P.

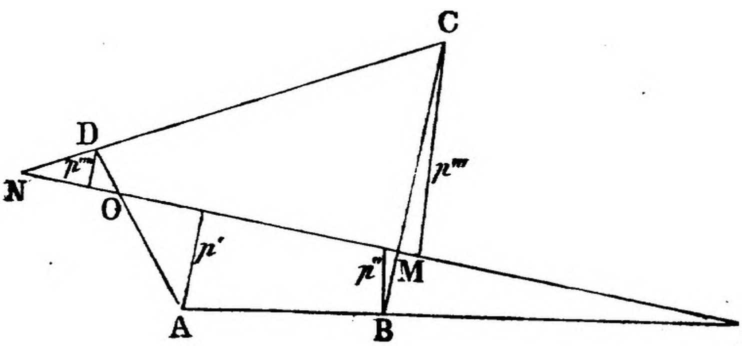
55. **Sol.**—Take any point G in the arc CD. Join CG, DG. From the centre O let fall a ⊥ OE on CD, and on OE describe a segment OFE containing an ∠ equal to CGD. Join OC. Bisect it in H. Through H draw HF ∥ to AB, cutting the segment OFE in F. Join OF, and through C draw CP ∥ to OF. P is the required point.

Dem.—Let CP intersect AB in K. Join PD, cutting AB in J. Produce FH to meet CP in I. Join EF, and produce it to



meet CP in L. Join CF, FJ. The points C, F, J are collinear; if not, let CF, FM be in a straight line. Now (III. xxii.) the $\angle$s CGD, CPD equal two right $\angle$s; $\therefore$ OFE, CPD equal two right $\angle$s, and OFE, OFL equal two right $\angle$s, $\therefore$ OFL = CPD; that is, CLE = CPD; hence EL is $\parallel$ to PD. Again, in the $\triangle$ COM, since CD is bisected in H, CM is bisected in (I. xl., Ex. 3); and similarly, in the $\triangle$ CDN, CN is bisected in F; $\therefore$ FN = FM, which is absurd; hence CF produced must pass through J, and CF = FJ. Now, in the $\triangle$ CJK, CJ is bisected in F, and OF is parallel to CP, $\therefore$ KJ is bisected in O; that is, OK = OJ.

56. Let ABCD be a polygon of four sides. Produce AB, CD to L, N, and draw a transversal LMON, cutting the four sides. From A, B, C, D let fall $\perp$s p', p'', p''', p'''' on LMON.



Now, since the Δ^s $Ap'L$, $Bp''L$ are equiangular,

$$\frac{AL}{BL} = \frac{p'}{p''} \text{ (IV.)}.$$

For the same reason,

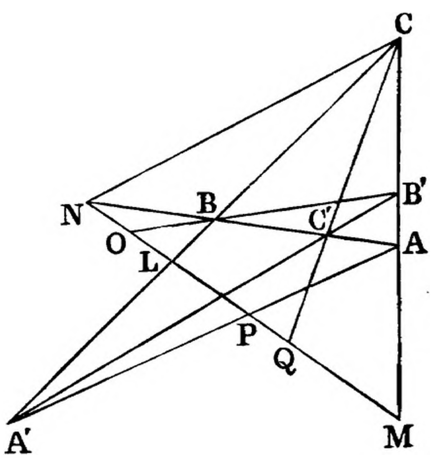
$$\frac{BM}{CM} = \frac{p''}{p'''}, \quad \frac{CN}{DN} = \frac{p'''}{p''}, \text{ and } \frac{DO}{AO} = \frac{p''''}{p'}.$$

Multiplying together, we get

$$\frac{AL \,.\, BM \,.\, CN \,.\, DO}{BL \,.\, CM \,.\, DM \,.\, AO} = \frac{p'p''p'''p''''}{p'p''p'''p''''}.$$

Hence AL . BM . CN . DO = BL . CM . DN . AO. And similarly for a figure of any number of sides.

57. Let the transversal LMN cut the sides of the Δ ABC in the points L, M, N. Bisect LN, NM, ML in O, P, Q. Join AP, OB, CQ, and produce them to meet the sides of the Δ ABC in A′, B′, C′, respectively. It is required to prove that the points A′, B′, C′ are collinear.



Dem.—The sides of the Δ AMN are cut by OB ; therefore

$$\frac{AB'}{B'M} \cdot \frac{MO}{ON} \cdot \frac{NB}{BA} = -1 \text{ (Ex. 5)}.$$

And since the Δ CLM is cut by OB′, $\dfrac{MB'}{B'C} \cdot \dfrac{CB}{BL} \cdot \dfrac{LO}{OM} = -1.$

Multiplying together, we have $\dfrac{AB'}{B'C} \cdot \dfrac{CB}{BA} \cdot \dfrac{NB}{BL} = 1$; interchange,

and $\frac{BC'}{C'A} \cdot \frac{AC}{CB} \cdot \frac{LC}{CM} = 1$; interchange again, and $\frac{CA'}{A'B} \cdot \frac{BA}{AC} \cdot \frac{MA}{AN} = 1$.

Multiply these results together, and we get

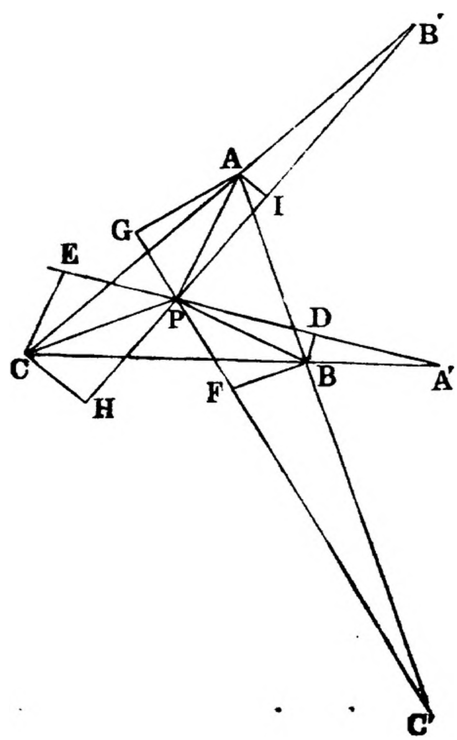
$$\frac{AB'}{B'C} \cdot \frac{BC'}{C'A} \cdot \frac{CA'}{A'B} \cdot \frac{NB}{BL} \cdot \frac{LC}{CM} \cdot \frac{MA}{AN} = 1;$$

but $\frac{NB}{BL} \cdot \frac{LC}{CM} \cdot \frac{MA}{AN} = -1$ (Ex. 5); $\therefore \frac{AB'}{B'C} \cdot \frac{BC'}{C'A} \cdot \frac{CA'}{A'B} = -1$.

And hence the points A′, B′, C′ are collinear.

58. Let ABC be the △. Join PA, PB, PC, and erect at P ⊥ˢ A′E B′H, C′G to PA, PB, PC, intersecting the sides BC, CA, AB, respectively in A′, B′, C′. It is required to show that the points A′, B′, C′ are collinear.

Dem.—From A, B, C let fall ⊥ˢ AG, AI on CG, B′H; BD, BF on A′E, C′G; CE, CH on A′E, B′H.



Now, because each of the ∠ˢ APA′, BPB′ is right, the ∠ API = BPD, and AIP = BDP; hence the △ˢ AIP, BDP are equiangular. In like manner, the △ˢ AGP, CEP are equiangular, and CPH, BPF are equiangular.

Again, since the $\triangle^s$ CA′E, BA′D are equiangular, $\dfrac{CA'}{A'B} = \dfrac{CE}{BD}$.

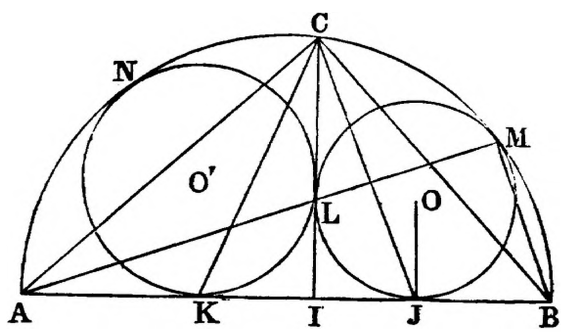
Similarly, $$\frac{AB'}{B'C} = \frac{AI}{CH}, \text{ and } \frac{BC'}{C'A} = \frac{BF}{AG};$$

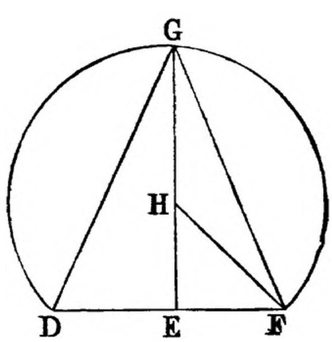
therefore $$\frac{CA' \cdot AB' \cdot BC'}{A'B \cdot B'C \cdot C'A} = \frac{CE \cdot AI \cdot BF}{BD \cdot CH \cdot AG};$$

hence
$$\frac{CA' \cdot AB' \cdot BC'}{A'B \cdot B'C \cdot C'A} = \frac{CE \cdot AI \cdot BF \cdot PB \cdot PC \cdot PA}{BD \cdot CH \cdot AG \cdot PB \cdot PC \cdot PA};$$

but AI . BP = BD . AP, since the $\triangle^s$ AIP, BDP are equiangular, and PA . CE = AG . PC; and PC . BF = PB . CH; ∴ CA′ . AB′ . BC′ = A′B . B′C . C′A. And hence (Ex. 4) the points A′, B′, C′ are collinear.

59. Let ACB be a given semicircle. It is required to divide it into two parts by a ⊥ on the diameter AB, so that the radii of the ⊙^{s} inscribed in them may have a given ratio DE : EF.





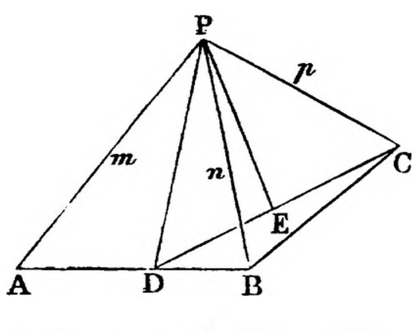
Sol.—On DF describe a segment containing an ∠ equal to half a right ∠. Erect EG ⊥ to DF. Join DG, FG. At the point F in FG draw FH, making the ∠ HFG = HGF. In the semicircle

draw BC, making the ∠ ABC = EFH. Let fall the ⊥ CI on AB. CI is the required line.

Dem.—In the figures CIBM, CIAN describe ⊙s, touching IB, IA, IC, and the arcs BC, AC in the points J, K, L, M, N. Let O, O′ be their centres. Join OJ, CJ, OL, O′L, AL, LM. The points A, L, M are collinear (III., Ex. 51). Join BM, AC, CK. Now the ∠ LIB is right, and LMB is right (III. xxxi.); ∴ ILMB is a cyclic quadrilateral; ∴ BA . AI = MA . AL; but BA . AI = AC^2 (I. xlvii., Ex. 1), and MA . AL = AJ^2 (III. xxxvi.); ∴ $AC^2 = AJ^2$; ∴ AC = AJ; ∴ the ∠ ACJ = AJC; ∴ ACJ = JBC + JCB; but ACI = IBC (viii.); ∴ ICJ = BCJ. In like manner, the ∠ ICK = ACK; hence the ∠ KCJ is half a right ∠. Now in the △s EHF, ICB the ∠ BIC = FEH, and IBC = EFH (const.); ∴ ICB = EHF; but ICB = 2 ICJ, and EHF = 2 EGF; ∴ ICJ = EGF, and CIJ = GEF; ∴ CJI = GFE; hence the △s CIJ, GEF are equiangular. And because the ∠ DGF = KCJ, and GFD = CJK; ∴ GDF = CKJ; hence the △s CJK, GFD are equiangular; ∴ KI : IJ : : DE : EF; but KI : IJ : : O′L : OL. Hence O′L : OL : : DE : EF.

Lemma.—If A, B, C be fixed points, and P a variable point, find the locus of P, if $mAP^2 + nBP^2 + pCP^2$ is given.

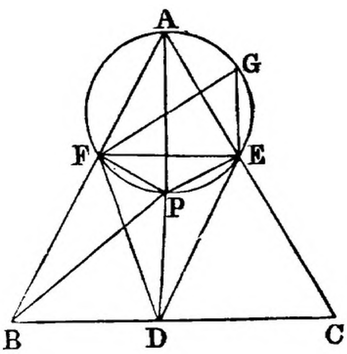
Sol.—Join AP, BP, CP, AB, BC. Divide AB in D, so that $mAD = nDB$. Join DP. Now $mAP^2 + nBP^2 = mAD^2 + nDB^2 + (m + n) DP^2$ (Book II., Ex. 12). Join DC, and divide it in E, so that $(m + n) DE = pEC$. Join EP; then $(m + n) DP^2 + pPC^2 = (m + n) DE^2 + pEC^2 + (m + n + p) EP^2$; add, and $mAP^2 + nBP^2$



$+ pPC^2 = mAD^2 + nDB^2 + (m + n) DE^2 + pEC^2 + (m + n + p) EP^2$; but $mAP^2 + nBP^2 + pPC^2$ is given (hyp.); ∴ $mAD^2 + nDB^2$ + &c., is given; but $mAD^2 + nDB^2$ is given, and $(m + n) DE^2$, and pEC^2 is given; ∴ $(m + n + p) EP^2$, and $(m + n + p)$ is given; ∴ EP^2

is given, ∴ EP is given, and E is a given point. Hence the locus of P is a circle, having E for centre and EP for radius.

60. **Dem.**—Let P be the point. From P let fall ⊥s PD, PE, PF on the sides of the △. Join DE, EF, FD, AP, BP, CP. Now because the ∠s AEP, AFP are right, AEPF is a cyclic quadrilateral; then AP is the diameter of the circumscribed ⊙. Draw FG, another diameter. Join GE. Now the ∠ FGE = FAE (III. xxi.); but FAE is a given ∠, ∴ FGE is a given ∠, and

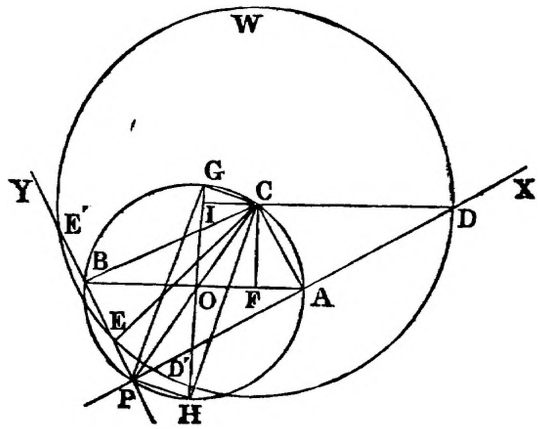


the ∠ FEG is given, being right; ∴ the △ FGE is given in species; hence $\frac{EF}{FG}$ is given; but FG = AP; ∴ $\frac{EF}{AP}$ is given; ∴ $\frac{EF^2}{AP^2}$ is given; let it be equal to m, then $EF^2 = mAP^2$. In like manner, $FD^2 = nBP^2$, and $DE^2 = pCP^2$; but $EF^2 + FD^2 + DE^2$ is given (hyp.); ∴ $mAP^2 + nBP^2 + pCP^2$ is given. And hence (*Lemma*) the locus of P is a circle.

61. Let the ⊙ W make given intercepts DD′, EE′ on two fixed lines PX, PY. It is required to prove that the rectangle CG . CH contained by the ⊥s from the centre C on the bisectors of the ∠s formed by the lines PX, PY is given.

Dem.—From C let fall ⊥s CA, CB on DD′, EE′. Join CD, CE. Now $AC^2 + AD^2 = CD^2$, and $BC^2 + BE^2 = CE^2$; ∴ $AC^2 + AD^2 = BC^2 + BE^2$; ∴ $AD^2 - BE^2 = BC^2 - AC^2$; but AD, BE are the halves of DD′, EE′ (III. iii.), and are given (hyp.); ∴ $BC^2 - AC^2$ is given. Now since the ∠s CAP, CBP are right, CAPB is a cyclic quadrilateral. Describe a ⊙ about it. Join AB; the line bisecting AB perpendicularly will be the diameter. Let it be GH. Join GP, HP; these are the internal and external bisectors of the ∠ EPD (III. xxx., Ex. 2). Join CP, CH,

and let fall $\perp^s$ CF, CI on AB, GH. Now $BC^2 = BF^2 + FC^2$, and $AC^2 = AF^2 + FC^2$; $\therefore BC^2 - AC^2 = BF^2 - FA^2$; but $BC^2 - AC^2$ is given; $\therefore BF^2 - FA^2$ is given; that is (BF + FA) (BF − FA) is given; but BF + FA = AB, and BF − FA = 2 OF; $\therefore$ AB . OF is given; that is, AB . CI is given. And because the $\angle$ APB is given, the ratio of AB to the diameter is given



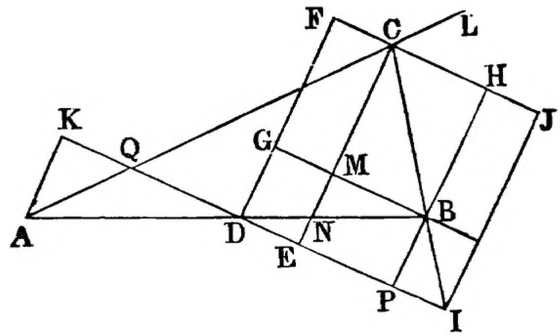
(Dem. of Ex. 60); that is, AB : GH is given; $\therefore$ the ratio AB . CI : GH . CI is given; but AB . CI is given; $\therefore$ GH . CI is given. And since the Δ^s GCH, ICH are equiangular, GH . CI = GC . CH. Hence GC . CH is given.

62. Let ABC be a Δ, whose base and the difference of whose base $\angle^s$ is given. Draw CE, CF, the internal and external bisectors of the vertical $\angle$. Bisect AB in D, and let fall $\perp^s$ DE, DF on CE, CF. It is required to prove that the rectangle DE . DF is given.

Dem.—Draw BG, BH ‖ to CF, DF. Produce CB to meet DE produced in I. Draw IJ ‖ to CE, and let fall a $\perp$ AK on ID produced.

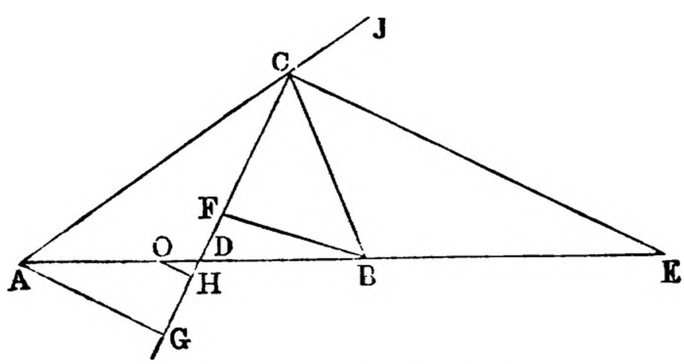
Now the $\angle$ NCB = ACN, and LCJ = ICJ; $\therefore$ the $\angle$ NCJ is right, and DFC is right; $\therefore$ DF is ‖ to CN; $\therefore$ FCMG is a parallelogram. Now the $\angle$ ANC = NCB + CBN, and BNC = NCA + CAN; $\therefore$ (ANC − BNC) = (NCB + CBN) − (NCA + CAN); but NCB = NCA; $\therefore$ (ANC − BNC) = (CBN − CAN); but (CBN − CAN) is given (hyp.), $\therefore$ (ANC − BNC) is given, and their sum is given; hence each is given; but DNE = BNC; $\therefore$ DNE is given, and DEN is right; $\therefore$ EDN is given; hence

the line IK is given in position; ∴ PB ⊥ to JK is given in position. And because the ∠ QCE = ICE, and CEQ = CEI, and CE common, EQ = EI, and the ∠ EQC = EIC; but EQC



= AQK; ∴ EIC or BIP = AQK, and AKQ = BPI, each being right, and the side AK = BP; ∴ KQ = IP. To each add QP, and we have KP = QI; hence (Ax. 7) KD = QE; ∴ KQ = DE; ∴ DE = IP; hence the figure GC = BJ; but BJ = BE (I. XLIII.); ∴ GC = BE; hence the rectangle DC = BD; that is, the rectangle DE . DF = BD; but BD is a given rectangle. Hence DE . DF is given.

63. Let ABC be the Δ. Bisect the ∠ ACB by CD. From A, B let fall ⊥s AG, BF on CD. Produce AC, and bisect the



∠ BCJ by CE, meeting AB produced in E. Bisect AB in O, and let fall a ⊥ OH on CD. It is required to prove that AG . FB = OH . CE.

Dem.—Now AD : DB : : AE : EB (III., Ex. 3); hence (Book V., Ex. 9) OD : OB : : OB : OE; that is, OD . OE = OB^2; but (II. III.) OD . OE = OD^2 + OD . DE, and (II. V.)

$OB^2 = AD \cdot DB + OD^2$; hence $OD \cdot DE = AD \cdot DB$; $\therefore$ AD : OD : : DE : DB; but AD : OD : : AG : OH, and DE : DB : : CE : FB; $\therefore$ AG : OH : : CE : FB. And hence $AG \cdot FB = OH \cdot CE$.

64. The rectangle contained by the perpendiculars from the extremities of the base on the external bisector of the vertical angle, is equal to the rectangle contained by the internal bisector and the perpendicular from the middle of the base on the external bisector.

Let ACB be the $\triangle$. Produce AC to J, and bisect the $\angle$ BCJ by ECG, meeting AB produced in E. From A, B let fall $\perp$s AG, BF on EG. Bisect the $\angle$ ACB by CD. Bisect AB in O, and let fall a $\perp$ OH on EG. It is required to prove that $AG \cdot BF = OH \cdot CD$.

Dem.—$AE \cdot EB + OB^2 = OE^2$ (II. VI.); but $OE^2 = OD \cdot OE + DE \cdot OE$ (II. II.); hence $AE \cdot EB + OB^2 = OD \cdot OE + DE \cdot OE$; hence $AE \cdot EB = DE \cdot OE$ (see Ex. 63); $\therefore$ AE : DE : : OE : EB. Hence, by similar triangles, AG : CD : : OH : BF; $\therefore$ $AG \cdot BF = OH \cdot CD$.

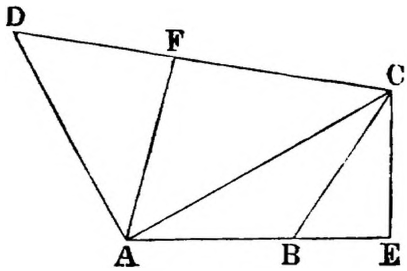
65. **Dem.**—From C let fall a $\perp$ CD on AB. Now the $\triangle$s ACD, BCD, ABC are similar (VIII.); then, if R, R$'$, ρ, are the radii of the $\odot$s inscribed in these $\triangle$s, AC, BC, AB are proportional to R, R$'$, ρ; but $AC^2 + BC^2 = AB^2$; $\therefore$ $R^2 + R'^2 = \rho^2$, and $\rho^2 = (s-c)^2$ (IV. IV., Ex. 14); that is, $R^2 + R'^2 = (s-c)^2$.

66. **Sol.**—Through A, C draw two parallel lines AF, CE; and through B, D draw two parallel lines BF, DE, meeting the $\|$ through A, C in F, E. Join EF, and produce it to meet AD in O.

Dem.—Because BF is $\|$ to DE, the $\triangle$s ODE, OBF are equiangular; hence OD : OB : : OE : OF; and since the $\triangle$s OCE, OAF are equiangular, OE : OF : : OC : OA, $\therefore$ OD : OB : : OC : OA. Hence $OA \cdot OD = OB \cdot OC$.

67. **Sol.**—Let a, b, c, d be the four sides. Find a fourth proportional to $(2ab + 2cd)$, $\{(c^2+d^2) - (a^2+b^2)\}$, and b. Let it be BE. Produce EB to A, so that $AB = a$. Erect EC $\perp$ to AE. With B as centre, and a radius equal to b, describe a $\odot$, cutting EC in C. Join BC, AC; and on AC describe a $\triangle$ ACD, having its sides CD, AD equal to c and d. ABCD is the required quadrilateral.

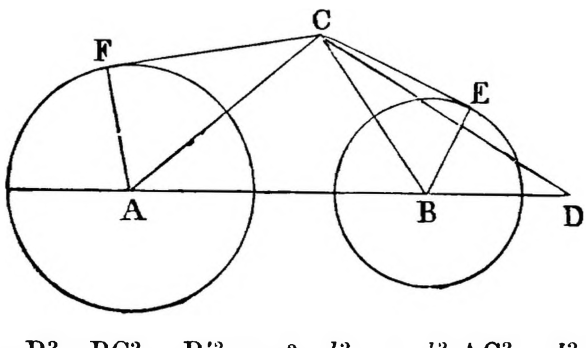
Dem.—From A let fall a $\perp$ AF on CD. Now because BE is a fourth proportional to $(2ab + 2cd)$, $\{(c^2 + d^2) - (a^2 + b^2)\}$, and b, we have $(2ab + 2cd)$ BE $= \{(c^2 + d^2) - (a^2 + b^2)\} b$. Now $AC^2 = AB^2 + BC^2 + 2 AB . BE$ (II. XII.); that is, $AC^2 = a^2 + b^2$



$+ 2a . BE$; and $AC^2 = c^2 + d^2 - 2c . DF$ (II. XIII.); $\therefore c^2 + d^2 - 2c . DF = a^2 + b^2 + 2a . BE$; $\therefore c^2 + d^2 - (a^2 + b^2) = 2a . BE + 2c . DF$; hence $(2ab + 2cd)$ BE $= (2a . BE + 2c . DF) b$; $\therefore 2cd . BE = 2bc . DF$; $\therefore d . BE = b . DF$; $\therefore d : DF :: b : BE$, that is, AD : DF :: BC : BE, and the $\angle$ AFD = BEC; $\therefore$ the $\triangle$s ADF, CBE are equiangular; $\therefore$ the $\angle$ ADF = CBE. To each add ABC, and we have the $\angle$s ADC, ABC equal to ABC, EBC; $\therefore$ ADC + ABC equal two right $\angle$s. Hence ABCD is a cyclic quadrilateral.

68. Let A, B be the centres of the $\odot$s. From a point C tangents CF, CE are drawn to the $\odot$s, so that CF : CE :: $a : b$. It is required to find the locus of C.

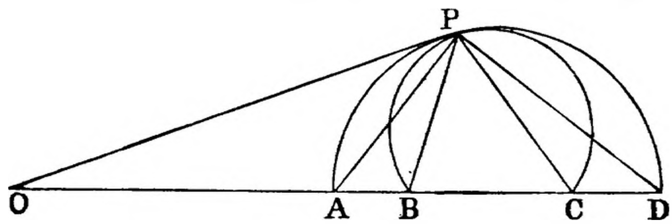
Sol.—Join AF, BE, AC, BC, and let the radii be denoted by R, R′. Now since CF : CE :: $a : b$, $CF^2 : CE^2 :: a^2 : b^2$; that



is, $AC^2 - R^2 : BC^2 - R'^2 :: a^2 : b^2$; $\therefore b^2 AC^2 - b^2 R^2 = a^2 BC^2 - a^2 R'^2$; $\therefore b^2 AC^2 - a^2 BC^2 = b^2 R^2 - a^2 R'^2$. Join AB, and produce it to D, and make AD : BD :: $a^2 : b^2$; then $b^2 AD = a^2 BD$. Now, joining CD, and putting b^2 for m, and a^2 for n, we have

(Book II., Ex. 13) $b^2\,AC^2 - a^2\,BC^2 = b^2\,AD^2 - a^2\,DB^2 + (b^2 - a^2)\,CD^2$, and $\therefore$ (Ax. 1) $b^2\,AD^2 - a^2\,DB^2 + (b^2 - a^2)\,CD^2 = b^2\,R^2 - a^2\,R'^2$; and transposing, we get $(a^2 - b^2)\,CD^2 = b^2\,(AD^2 - R^2) - a^2\,(DB^2 - R'^2)$; $\therefore$ $(b^2 - a^2)\,CD^2$ is given; $\therefore$ CD is given, and the point D is given. Hence the locus of C is a circle.

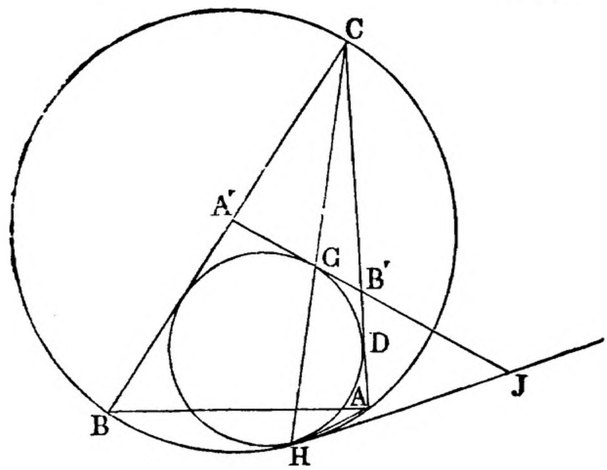
69. **Sol.**—Describe ⊙s about the △s APD, BPC. Draw OP a tangent to the ⊙ APD, meeting DA produced in O. Now



the ∠ OPA = PDA (III. xxxii.), and the ∠ APB = CPD (hyp.); $\therefore$ the ∠ OPB = ADP + CPD = ACP; hence OP touches the ⊙ BPC. Now (III. xxxvi.) OA . OD = OP^2, and OB . OC = OP^2, $\therefore$ OA . OD = OP . OC; $\therefore$ O is a given point (Ex. 66), and A, D are given points; $\therefore$ OA . OD is given; $\therefore$ OP^2 is given; $\therefore$ OP is given. Hence the locus of P is a ⊙, having O as centre and OP as radius.

70. If a ⊙ ACB be circumscribed to a △, and a ⊙ GBH be inscribed, touching the sides AC, BC in D, F, and the circumscribed ⊙ in H. It is required to prove that CD is a fourth proportional to the semi-perimeter of the △ ABC, and the sides CA, CB.

Dem.—Join CH, and draw HJ a tangent to the ⊙ ABC; at G draw a tangent A′J to the ⊙ DFH. Join AH.



Because JG = JH, the ∠ JHG = JGH; but JGH = GBC

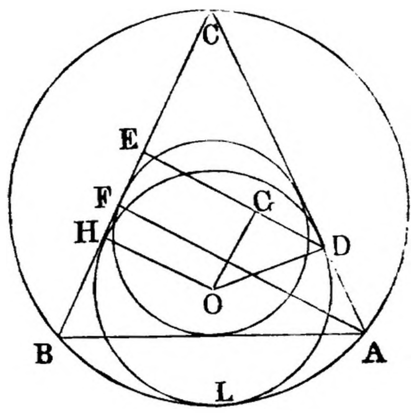
+ B′CG, ∴ JHG = GB′C + B′CG, and AHJ = GCB′ (III. XXXII.); ∴ GHA = GB′C. To each add GB′A, and we have GB′C + GB′A = GB′A + GHA, ∴ GB′A + GHA equal two right ∠s; hence GB′AH is a cyclic quadrilateral, and therefore HC . CG = AC . CB′; but HC . CG = CD^2 (III. XXXVI.); ∴ AC . CB′ = CD^2. Again, the ∠ CHA = A′B′C; but CHA = CBA (III. XXI.), ∴ CBA = CB′A′, and the ∠ A′CB is common, ∴ the △s ABC, A′B′C are equiangular; and, denoting their semiperimeters by s, s', we have (XX., Cor. 1) $s : s' :: BC : B'C$, ∴ $s : s' . CA :: BC : B'C . CA$; that is, $s : s' . CA :: BC : CD^2$; but $CD^2 = s'^2$ (IV. IV., Ex. 4); ∴ $s : s' . CA :: BC : s'^2$. Hence $s : CA :: BC : s'$; or, $s : CA :: CB : CD$.

71. It is an obvious modification of 70.

73. Let the sides AC, BC of the △ ABC, circumscribed to a given ⊙, be given in position, but the third side AB variable. About ABC describe a ⊙. It is required to prove that the ⊙ about ABC touches a fixed circle.

Dem.—Describe a ⊙, touching the sides AC, BC in D, H, and the ⊙ about ABC in L. Let O be its centre. Join OD, OH. Let fall a ⊥ AF on BC. Draw DE ∥ to AF, and let fall a ⊥ OG on DE.

Now $s : CB :: CA : CD$ (Ex. 70); but $CA : CD :: AF : DE$; therefore $s : CB :: AF : DE$, ∴ $s . DE = CB . AF$ = twice the area



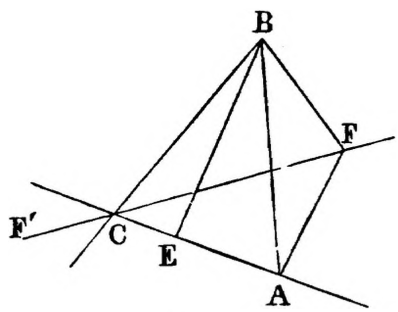
of the △ ABC = $2\,rs$ (IV. IV., Ex. 9), ∴ DE = $2\,r$; but $2\,r$ is given, ∴ DE is given; and because the ∠ ECD is given (hyp.), and the ∠ E is right, the △ ECD is given in species, ∴ the ratio ED : DC is given; but ED is given, ∴ DC is given; ∴ D is a given point.

Again, because the $\angle$ ODC is right, and $\therefore$ = ECD + CDE, $\therefore$ ODG = ECD. Hence ODG is given, and OGD is right, $\therefore$ the $\triangle$ OGD is given in species, $\therefore$ the ratio OD : DG is given; but OD = OH = GE, $\therefore$ the ratio EG : GD is given; but ED is given; $\therefore$ EG, that is OD, is given, and the point D has been shown to be given. Hence the $\odot$, with O as centre, and OD as radius, is a fixed $\odot$, and the $\odot$ about ABC touches it in L.

74. Let AC, BC be the two sides given in position.

Sol.—Bisect the $\angle$ ACB by FF′. In CF find a point F, such that $CF^2 = CA . CB$. F is one of the required points.

Dem.—Join AF, BF, and let fall a $\perp$ BE on AC. Now



because the area of the $\triangle$ ACB is given, CA . EB is given; and since the $\angle$ BCE is given, and the $\angle$ BEC is right, the $\triangle$ BCE is given in species, $\therefore$ the ratio CB : BE is given, $\therefore$ the ratio CB . CA : BE . CA is given; but CB . CA = CF^2 (const.), and BE . CA is given; $\therefore$ CF^2 is given, $\therefore$ CF is given, and $\therefore$ F is a given point. Again, because CA . CB = CF^2, CA : CF : : CF : CB, and the $\angle$ ACF = BCF, $\therefore$ (VI.) the $\angle$ CFA = CBF. To each add the sum of the $\angle$s CFB, BCF, and we have the sum of the $\angle$s of the $\triangle$ CBF equal to the $\angle$s AFB and BCF, $\therefore$ AFB and BCF are equal to two right $\angle$s; but the $\angle$ BCF is given, $\therefore$ AFB is given. Hence the base AB subtends a constant $\angle$ at a given point F. In like manner it can be shown that it subtends a constant $\angle$ at F′, constructed by making CF′ = FC.

75. Let ABCD be the cyclic quadrilateral. (See Diagram, Ex. 67.)

Dem.—Draw the diagonal AC. Produce AB, and let fall the $\perp$s AF, CE on AB, CD.

Now, since the sides AB, BC, CD, DA are denoted by a, b, c, d, we have (II. XII.) $AC^2 = a^2 + b^2 + 2a \,.\, BE$, and (II. XIII.) $AC^2 = c^2 + d^2 - 2c \,.\, DF$; $\therefore c^2 + d^2 - 2c \,.\, DF = a^2 + b^2 + 2a \,.\, BE$; $\therefore c^2 + d^2 - (a^2 + b^2) = 2a \,.\, BE + 2c \,.\, DF$; and because the $\triangle^s$ BCE, ADF are equiangular, BC : BE :: AD : DF; that is, b : BE :: d : DF, $\therefore b \,.\, DF = d \,.\, BE$, $\therefore DF = \frac{d}{b} \,.\, BE$; and

hence we have $c^2 + d^2 - (a^2 + b^2) = 2a \,.\, BE + \frac{2cd}{b} \,.\, BE$

$$= \frac{2(ab + cd)}{b} \,.\, BE; \ \therefore BE = \frac{b\{(c^2 + d^2) - (a^2 + b^2)\}}{2(ab + cd)}.$$

Again, $CE^2 = BC^2 - BE^2 = b^2 - \frac{b^2\{(c^2 + d^2) - (a^2 + b^2)\}^2}{4\{(ab + cd)^2\}}$

$$= b^2\left\{1 - \frac{\{(c^2 + d^2) - (a^2 + b^2)\}^2}{4(ab + cd)^2}\right\}$$

$$= b^2 \frac{\{4(ab + cd)^2 - \{(c^2 + d^2) - (a^2 + b^2)\}^2}{4(ab + cd)^2}$$

$$= b^2 \frac{\{(c + d)^2 - (a - b)^2\}\{(a + b)^2 - (c - d)^2\}}{4(ab + cd)^2}$$

$$= b^2 \frac{\{(c + d + a - b)(c + d - a + b)(a + b + c - d)(a + b - c + d)\}}{4(ab + cd)^2}.$$

Hence, putting $(a + b + c + d) = 2s$, and substituting, we get]

$$CE^2 = \frac{16\,b^2 \,.\, (s - a)(s - b)(s - c)(s - d)}{4(ab + cd)^2};$$

$$\therefore CE = \frac{2b\sqrt{(s - a)(s - b)(s - c)(s - d)}}{ab + cd}.$$

Now $AB = a$, and $AB \,.\, CE = 2 \triangle ABC$.

$$\therefore 2ABC = \frac{2ab\sqrt{(s - a)(s - b)(s - c)(s - d)}}{(ab + cd)};$$

$$\therefore ABC = \frac{ab\sqrt{(s - a)(s - b)(s - c)(s - d)}}{ab + cd}.$$

Similarly, $ACD = \frac{cd\sqrt{(s - a)(s - b)(s - c)(s - d)}}{(ab + cd)}.$

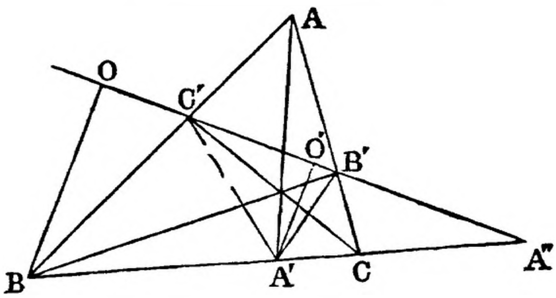
Hence the quadrilateral ABCD

$$= \frac{(ab + cd)\sqrt{(s-a)(s-b)(s-c)(s-d)}}{(ab + cd)}$$

$$= \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

76. **Dem.**—Produce BC, C′B′ to meet in A″. Let fall ⊥s A′O′, BO on A″C′.

Now AB′. BC′. CA′ = A′B . B′C . C′A (Ex. 4), and AB′.BC′.CA″ = A″B . B′C . C′A (Ex. 5). Divide, and we get $\frac{CA'}{CA''} = \frac{A'B}{A''B}$; ∴ A″B . A′C = A″C . A′B, ∴ A″B . A′C + A″C . A′B = 2 A″B . A′C; that is ("Sequel," Book II., Prop. VII.), A″A′. CB = 2 A″B . A′C.



Now the △ ABC : ABB′ : : AC : AB′ (I.), and ABB′ : BC′B′ : : AB : BC′, and BC′B′ : A′B′C′ : : BO : A′O′ : : BA″ : A′A″; that is, since A″A′. CB = 2 A″B . A′C : : BC : 2 A′C; ∴ △ ABC : A′B′C′ : : AB . BC . CA : 2 AB′. BC′. CA′.

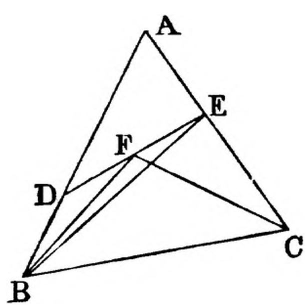
77. **Dem.**—Draw the diameter AE. Join BE, and let fall a ⊥ AD on BC. Now (XVII., Ex. 5) AE . AD = AB . AC; ∴ AE . AD . BC = AB . BC . CA; but AD . BC = twice the △ ABC; ∴ 2 AE . ABC = AB . BC . CA; hence (Ex. 76) ABC : A′B′C′ :: 2AE . ABC : 2AB′. BC′. CA′, ∴ 1 : A′B′C′ :: AE : AB′. BC′. CA′; ∴ AE . A′B′C′ = AB′. BC′. CA′; and hence

$$AE = \frac{AB' . BC' . CA'}{A'B'C'}.$$

78. **Dem.**—Let the sides of the quadrilateral be denoted by a, b, c, d. Now (III. XVII., Ex. 3) $(a + c) = (b + d)$; ∴ $2(a + c) = (a + b + c + d)$. Hence, putting $(a + b + c + d) = 2s$, we have

$2(a+c) = 2s$, $\therefore (a+c) = s$; $\therefore a = (s-c)$. Similarly, $b = (s-d)$, $c = (s-a)$, $d = (s-b)$; and (Ex. 74), we have area of quadrilateral $= \sqrt{(s-a(s-b)(s-c)(s-d)}$; $\therefore$ area $= \sqrt{abcd}$. Hence the square of the area $= abcd$.

79. **Dem.**—Join BF, CF, BE. Let the ratio BD : AD be denoted by m, n. Now the $\triangle$ ABC : ABE :: AC : AE (I.) :: AB : BD (hyp.); that is, as $(m+n) : m$, and ABE : BDE :: $(m+n)$ m,



and BDE : BDF :: $(m+n) : m$. Multiplying together, we have ABC : BDF :: $(m+n)^3 : m^3$; hence BDF $= \dfrac{\text{ABC} \cdot m^3}{(m+n)}$. In like manner ECF $= \dfrac{\text{ABC} \cdot n^3}{(m+n)^3}$. Again (XXIII., Ex. 1), ABC : ADE :: $(m+n)^2 : mn$; $\therefore$ ADE $= \dfrac{\text{ABC} \cdot mn}{(m+n)^2}$.

Now the $\triangle$ BFC = ABC − BDF − CEF − ADE = ABC

$$\left\{1 - \frac{m^3}{(m+n)^3} - \frac{n^3}{(m+n)^2} - \frac{mn}{(m+n)^2}\right\} = \text{ABC}\,\frac{2mn}{(m+n)^2}.$$

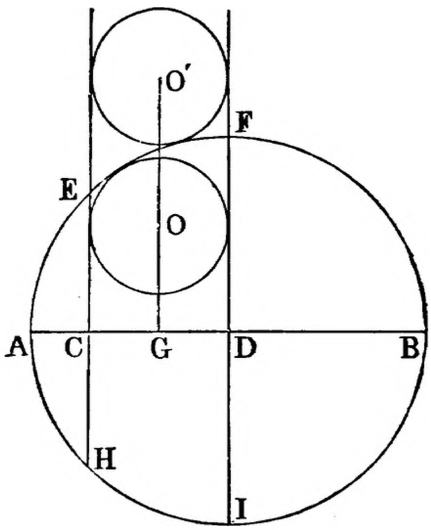
Hence the $\triangle$ BFC = twice the $\triangle$ ADE.

80. Let ABCD be a quadrilateral. Join AC, BD, and bisect them in E, F. Through E, F draw EG, FG parallel respectively to BD, AC. Bisect AD, CD in H, I. Join GH, GI. It is required to prove GIDH $= \frac{1}{4}$ ABCD.

Dem.—Join HF, IF, IH. Now, because AD, BD are bisected in H, F, HF is $\parallel$ to AB, and the $\triangle$ DHF $= \frac{1}{4}$ ADB (I. XL., Ex. 2). In like manner, DFI $= \frac{1}{4}$ DBC; $\therefore$ DHFI $= \frac{1}{4}$ ABCD. Again, HI is = to AC, and FG is $\parallel$ to AC; $\therefore$ HI is $\parallel$ to FG;

$\therefore$ (I. xxxvii.) the Δ HFI = HGI. To each add HDI, and HDIF = HGID; $\therefore$ HGID = $\frac{1}{4}$ ABCD. In like manner, if we bisect BC in J, and join GJ, GICJ = $\frac{1}{4}$ ABCD, &c.

81. **Dem.**—Let O, O′ be the centres. Join OO′, and produce it to meet AB in G; O′G is evidently perpendicular to AB. Complete the circle on AB, and produce EC, FD to meet it again in H, I.



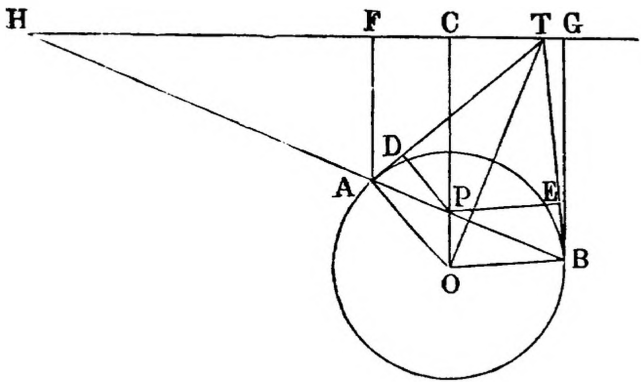
Now AC . DB = OG^2 (xiii., Ex. 5), and AD . CB = $O'G^2$ (xiii., Ex. 7); hence AC . CB . AD . DB = $OG^2 . O'G^2$; but AC . CB = CE^2, and AD . DB = DF^2; therefore $CE^2 . DF^2 = OG^2 . O'G^2$. And hence CE . DF = OG . O′G.

82. Let ABCDE be the inscribed regular polygon. Take any point P in the circumference. Join PA, PB, PC, PD, PE, and let those lines be denoted by ρ_1, ρ_2, ρ_3, ρ_4, ρ_5. It is required to prove that $\rho_1 + \rho_3 + \rho_5 = \rho_2 + \rho_4$.

Dem.—Join BD. Let the sides of the polygon be denoted by s, and the diagonals by d. Now, considering the polygon ABDP formed by ρ_1, ρ_2, ρ_4, we have (xvii., Ex. 13) $\rho_1 d + \rho_4 s = \rho_2 d$. Similarly, we have $\rho_1 d = \rho_2 s + \rho_4 s$, and $\rho_5 d + \rho_2 s = \rho_4 d$. Adding, we get $(\rho_1 + \rho_3 + \rho_5)d = (\rho_2 + \rho_4)d$. Hence $\rho_1 + \rho_3 + \rho_5 = \rho_2 + \rho_4$.

83. Let O be the centre of the given $\odot$; P the given point; AB any chord passing through P; PD, PE perpendiculars on the

tangents AT, BT. It is required to prove that the sum of the reciprocals of PD, PE is constant.



Dem.—Join OP, produce it, and from T let fall the ⊥ TC on OP produced. Produce AB to meet CT in H, and let fall the ⊥s AF, BG.

Now ("Sequel," Book III., Prop. XXVIII.) CT is the polar of P, and AT is the polar of A. Hence ("Sequel," Book III., Prop. XXVII.), since PD and AF are perpendiculars on the polars,

$$OA : OP : \; AF : DP; \; \therefore \frac{1}{DP} = \frac{OA}{OP} \cdot \frac{1}{AF}.$$

In like manner, $$\frac{1}{PE} = \frac{OB}{OP} \cdot \frac{1}{BG}.$$

Hence, denoting the radius of the circle by r, and the distance OP by d, we have

$$\frac{1}{PD} + \frac{1}{PE} = \frac{r}{d} \cdot \left(\frac{1}{AF} + \frac{1}{BG}\right)$$

Again, since P is the pole of the line GH, the line HB is cut harmonically; ∴ HP is a harmonic mean between HA and HB; but AF, PC, BG are proportional to HA, HP, HB; hence PC is a harmonic mean between AF and BG;

$$\therefore \frac{2}{PC} = \frac{1}{AF} + \frac{1}{BG}; \; \therefore \frac{1}{PD} + \frac{1}{PE} = \frac{r}{d} \cdot \frac{2}{PC}.$$

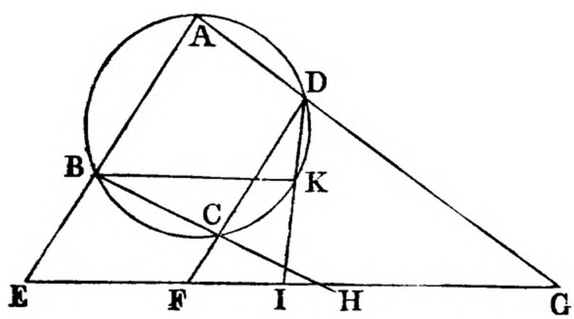
Hence the proposition is proved.

84. Let ABC be a cyclic quadrilateral, whose sides AB, CD, AD pass through three collinear points E, F, G. Join BC, and

produce it to meet EG in H. It is required to prove that H is a fixed point.

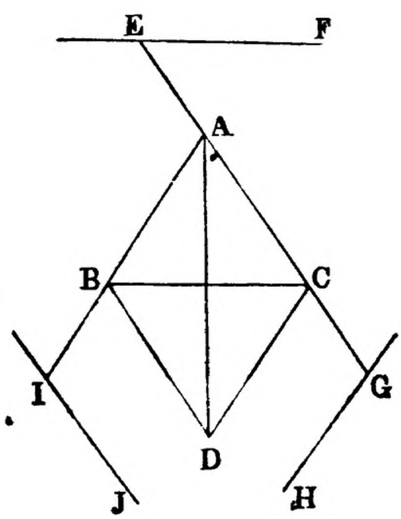
Dem.—Through B draw BK ∥ to EG. Join DK, and produce it to meet EG.

Now the ∠s ADK, ABK equal two right ∠s (hyp.); but ABK = AEG (I. xxix.), ∴ AEI and ADI are equal to two right ∠s; hence AEID is a cyclic quadrilateral; ∴ EG . GI = DG . GA; but DG . GA is given; ∴ EG . GI is given, and EG is given, ∴ GI



is given; ∴ I is a given point. Again, the ∠ IDF = KBC (III. xxi.); but KBC = CHF; ∴ IDF = CHF, and ∴ the points D, C, I, H are concyclic; hence DF . FC = FI . FH; but DF . FC is given; ∴ FI . FH is given, and FI is given; ∴ FH is given. And hence H is a given point.

85. (1) Suppose the polygon to be a △. Let BCD be a △,



whose sides are parallel to three given lines EF, GH, IJ; and

let the loci of its angular points B, C, be right lines AB, AC. It is required to prove that the locus of D is a right line.

Dem.—Join AD. Produce CA to meet EF in F.

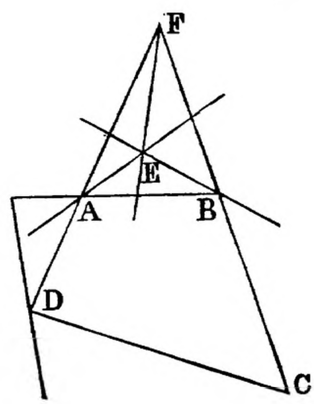
Now the $\angle$ BCA = FEA; $\therefore$ BCA is a given $\angle$, and the $\angle$ BAC is given, since the lines AB, AC are given in position; hence the $\triangle$ ACB is given in species; $\therefore$ the ratio AC : CB is given.

Similarly, the ratio BC : CD is given; $\therefore$ the ratio AC : CD is given, and the $\angle$ ACD is given; hence the $\triangle$ ACD is given in species; $\therefore$ the $\angle$ CAD is given, and the line AC is given in position; therefore the line AD is given in position. Hence the line AD is the locus of D.

(2) Let the polygon be the quadrilateral ABCD, having its sides parallel to four given lines, and the loci of the $\angle^s$ A, B, D right lines.

Dem.—Let the loci of A, B meet in E. Produce DA, CB to meet in F. Join EF.

Now AFB is a $\triangle$, whose three sides are parallel to three given lines, and the loci of A, B are right lines. Hence (1) the locus of F is the line EF, which is therefore given in position.



Again, DFC is a $\triangle$, having its sides parallel to three given lines, and having straight lines for the loci of D and F. Hence—(1) the locus of C is a right line. In like manner it can be proved for a figure of any number of sides.

86. Let BAC be a $\triangle$ whose vertical $\angle$ BAC, and its bisector AD, are given. It is required to prove that $\frac{1}{AC} + \frac{1}{AB}$ is given.

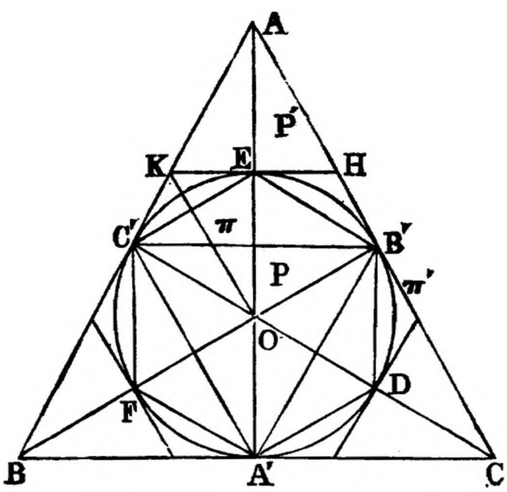
Dem.—Describe a $\odot$ about ABC. Produce AD to meet the circumference in E. Join EC, and let fall a $\perp$ EF on AB.

Now AF = $\frac{1}{2}$ (AB + AC) (III. xxx., Ex. 4). And since the $\angle$ BAC is bisected by AE, FAE is a given $\angle$, and the $\angle$ AFE is right; $\therefore$ the $\triangle$ AFE is given in species; $\therefore \frac{AF}{AE}$ is given; $\therefore \frac{2AF}{AE}$, that is, $\frac{AB + AC}{AE}$ is given, and AD is given (hyp.); $\therefore \frac{AB + AC}{AD . AE}$ is given. Again, the $\angle$ ABC = AEC (III. xxi.), and BAD = CAE; $\therefore$ the $\triangle$s BAD, CAE are equiangular; $\therefore$ AB : AD : : AE : AC; hence AB . AC = AD . AE; $\therefore \frac{AB + AC}{AB . AC}$ is given; that is, $\frac{AB}{AB . AC} + \frac{AC}{AB . AC}$ is given. Hence $\frac{1}{AC} + \frac{1}{AB}$ is given.

87. (1) Let the polygons be the $\triangle$s A′B′C′, ABC. Bisect the arcs A′B′, B′C′, C′A′ in the points D, E, F. Join A′D, DB′, B′E, EC′, C′F, FA′. This hexagon is the corresponding polygon of double the number of sides. It is required to prove that the hexagon is a geometric mean between the $\triangle$s ABC, A′B′C′.

Dem.—Join AO, A′O, BO, B′O, CO, C′O. Let OC intersect A′B′ in N.

Now we have the $\triangle$ OB′C : OB′D : : OC : OD (I.), and OB′D : OB′N : : OD : ON; but OC : OD : : OD : ON; hence OB′C : OB′D : : OB′D : OB′N; that is, the $\triangle$ OB′D is a geometric mean between the $\triangle$s OB′C, OB′N; but the hexagon is six times OB′D, ABC six times OB′C, and A′B′C′ six times OB′N. Hence, denoting the areas by P, P′, π, we see that π is a geometric mean between P and P′.

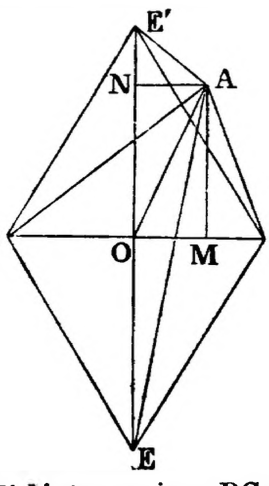


(2) At the points D, E, F draw tangents to the $\odot$; the figure,

whose sides are those tangents, and the parts cut off by them from the sides AC, CB, BA, is a circumscribed polygon of double the number of sides.

Dem.—Join OK. Now, since A′C′ is parallel to OK, AO : OA′ : : AK : KC; but OA′ = OE, ∴ AO : EO : : AK : KC′. Again (I.), the △ AOC′ : EOC′ : : AO : EO, and AKE : EKC′ : : AK : KC′; ∴ AOC′ : EOC′ : : AKE : EKC′. Now consider the figures AOC′, OEKC′, and OEC′. AOC′ is the first, OEC′ the third, and OEKC′ the second; and we have shown AOC′ : EOC′ : : AKE : EKC′; that is, the 1st : 3rd : : (1st − 2nd) : (2nd − 3rd); ∴ OEKC′ is a harmonic mean between OEC′ and AOC′; but OEKC′ is $\frac{1}{6}$ of π', OEC′ is $\frac{1}{6}$ of π, and AOC′ $\frac{1}{6}$ of P′. Hence π' is a harmonic mean between π and P′. In the same manner the proposition may be proved for a polygon of any number of sides.

88. *Lemma.*—If upon the base BC of a △ ABC two equilateral △s BCE, BCE′ be described on opposite sides, and their vertices E, E′ joined to A, then (1) if S denote the area of ABC, $AE'^2 - AE'^2 = 4\,S\sqrt{3}$; (2) $AE^2 + AE'^2 = AB^2 + CA^2$.



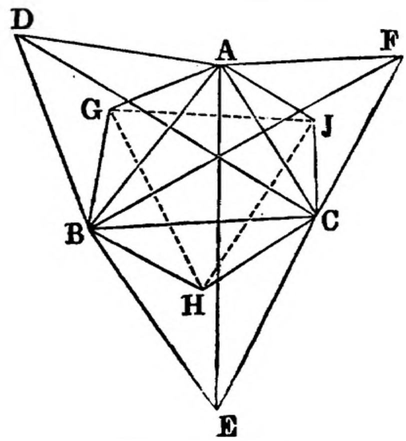
(1) **Dem.**—Join EE′, intersecting BC in O. Join AO, and draw AM, AN perpendicular to BC, EE′. Now $AE^2 - AE'^2 = EN^2 - NE'^2 = 4EO \,.\, ON = 4\sqrt{3} \,.\, OC \,.\, ON$; but OC . ON = area of the △ ABC = S; ∴ $AE^2 - AE'^2 = 4\sqrt{3} \,.\, S$.

(2) $AE^2 + AE'^2 = 2\,AO^2 + 2\,AE^2 = 2\,AO^2 + 6\,OC^2$. Again, $AB^2 + AC^2 = 2AO^2 + 2\,OC^2$, and $BC^2 = 4\,OC^2$; ∴ $AB^2 + BC^2 + CA^2 = 2\,AO^2 + 6\,OC^2$. Hence $AE^2 + AE'^2 = AB^2 + BC^2 + CA^2$.

Let ABC be the △; G, H, J the circumcentres of the equi-

lateral $\triangle$s constructed outwards on its sides. Join AG, AJ; BG, BH; CJ, CH; and GH, HJ, JG.

Now the $\angle$ EBH = ABG, because each is half an $\angle$ of an



equilateral $\triangle$; to each add HBA, and we have the $\angle$ EBA = HBG.

Again, $EB^2 = 3\,BH^2$, and $AB^2 = 3\,BG^2$; $\therefore$ EB : BA : : BH : HG. Hence the $\triangle$s EBA, HBG are equiangular; $\therefore$ $EB^2 : EA^2 :: BH^2 : HG^2$; but $EB^2 = 3\,BH^2$; $\therefore EA^2 = 3\,GH^2$.

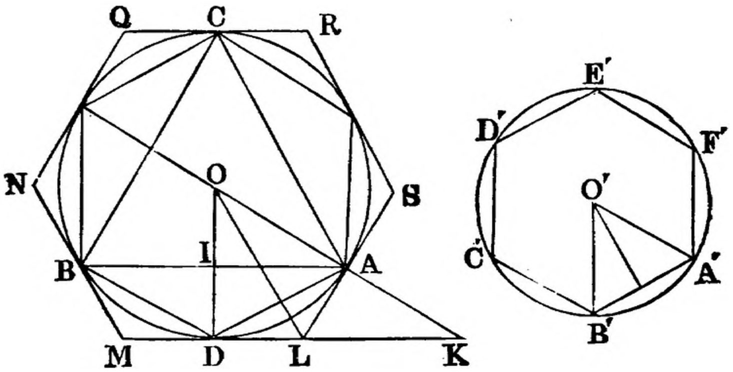
In like manner it may be proved, if G′, H′, J′ be the circumcentres of the equilateral $\triangle$s constructed inwards on the sides of ABC, that $AE'^2 = 3\,G'H'^2$. Hence $AE^2 - AE'^2 = 3\,(GH^2 - G'H'^2)$.

Again, denoting the areas of the equilateral $\triangle$s GHJ, G′H′J′ by Σ, Σ', we have $\Sigma = \dfrac{GH^2\sqrt{3}}{4}$, $\Sigma' = \dfrac{G'H'^2\sqrt{3}}{4}$; $\therefore 4\sqrt{3}\,(\Sigma - \Sigma') = 3\,(GH^2 - G'H'^2)$; but $4\sqrt{3}\,S = AE^2 - AE'^2$ (*Lemma*); $\therefore \Sigma - \Sigma' = S$.

89. From last demonstration we have $AE^2 + AE'^2 = 3(GH^2 + G'H'^2)$; but $AE^2 + AE'^2 = AB^2 + BC^2 + CA^2$ (*Lemma*); $\therefore 3\,(GH^2 + G'H'^2) = AB^2 + BC^2 + CA^2$, or the sum of the squares of the sides of the two equilateral $\triangle$s GHJ, G′H′J′ is equal to the sum of the squares of the sides of the $\triangle$ ABC.

90. (1) Let ABC be a regular polygon of three sides, the radii of whose circumscribed and inscribed $\odot$s are denoted by R, r; A′B′C′D′E′F′ a regular polygon of the same area, and double the number of sides, the radii of whose circumscribed and inscribed $\odot$s are R′, r'. It is required to prove that $R' = \sqrt{Rr}$.

Dem.—Join OA (R), O′A′ (R′), and let fall a ⊥ OI (r) on AB. Produce OI to meet the ⊙ in D. Join AD, B′D, O′B′. Now (I.) the △ OAD : OAI : : OD : OI; that is, as R : r; but OAI



= O′A′B′; ∴ OAD : O′A′B′ : : R : r; but (XIX.) OAD : O′A′B′ : : OA^2 : $O'A'^2$; that is, as $R^2 : R'^2$; hence $R : r :: R^2 : R'^2$; ∴ $RR'^2 = R^2r$; ∴ $R'^2 = Rr$. And hence $R' = \sqrt{Rr}$.

(2) It is required to prove that $r' = \dfrac{\sqrt{r\,(R + r)}}{2}$.

Dem.—Join OA. Let fall a ⊥ OI on AB, and produce it to meet the ⊙ in D. Through D draw a tangent MK, and produce OA to meet it. Through A, B draw tangents LS, MN. Bisect the arcs AC, BC, and at the points of bisection and at C draw tangents SR, NQ, RQ. Join OL.

Now OA : OI : : OK : OD; but OK : OD : : KL : LD; ∴ OA : OI : : KL : LD; that is, R : r : : KL : LD; ∴ $(R + r) : r$: : KD : LD; and KD : LD : : △ OKD : OLD; ∴ $(R + r) : r$: : OKD : OLD; ∴ $(R + r)\,r : 2r^2$: : OKD : 2 OLD, or OALD. Again (XIX.), $r^2 : R^2$: : OAI : OKD. Hence, multiplying these proportions, we get $(R + r)\,r : 2\,R^2$: : OAI : OALD; but OAI : OALD : : ABC : LMNQRS; that is, OAI : OALD : : A′B′C′D′E′F′ : LMNQRS; that is, as $r'^2 : R^2$; ∴ $(R + r)\,r : 2\,R^2 :: r'^2 : R^2$;

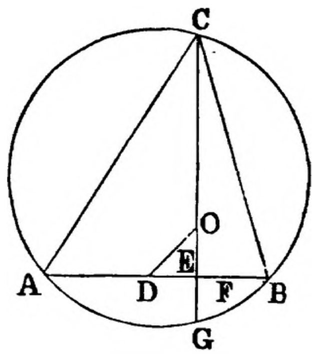
∴ $(R + r)\,r = 2\,r'^2$. Hence $r' = \sqrt{\dfrac{(R + r)\,r}{2}}$.

In the same way the proposition may be proved for a polygon of any number of sides.

91. **Dem.**—Let fall a ⊥ CE on AB; then CE = AB (hyp.). Describe a ⊙ about ABC, and produce CE to meet it in G. Let O be the orthocentre. Cut off BF = OE. Bisect AB in D. Join

OD. Now since BF = OE, and AB = CE, $\therefore$ AF = CO. Now AF . FB + DF² = DB² (II. v.); that is, CO . OE + DF² = DB².

Again, AE . EB + DE² = DB²; $\therefore$ CE . EG + DE² = DB²; $\therefore$ CE . EO + DE² = DB²; $\therefore$ (CO + OE) EO + DE² = DB²;

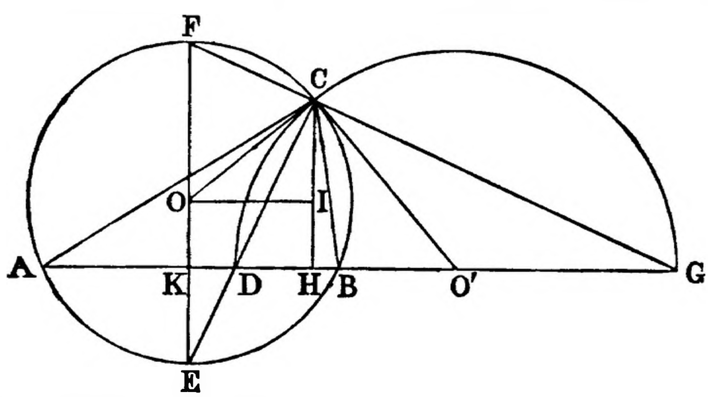


$\therefore$ CO . EO + EO² + DE² = DB²; $\therefore$ CO . EO + OD² = DB²; $\therefore$ OD² = DF²; $\therefore$ OD = DF, and OE = FB (const.) Hence OD + OE = DF + FB = DB.

92. Let ABC be any $\triangle$. Describe a $\odot$ about ABC. Draw the diameter EF $\perp$ to AB. Join CE, CF; these are the internal and external bisectors of the $\angle$ ACB. Produce FC, AB to meet in G. Let fall a $\perp$ CH on AB; it is evident that the $\odot$ on DG as diameter will be the locus of C when the base and ratio of the sides are given. Let O, O′ be the centres. Join OC, O′C. It is required to prove that AC² − CB² : 4 times area : : OC : O′C.

Dem.—Through O draw OI ∥ to AB.

Now the $\angle$ FOC = 2 FEC (III. xx.); but FOC = OCI :.

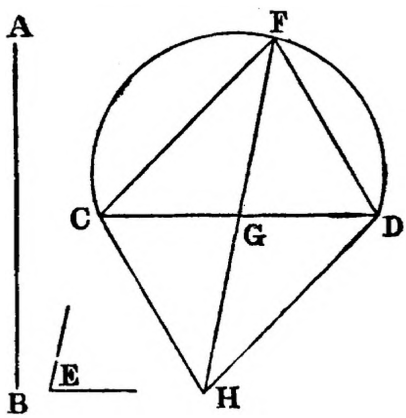


$\therefore$ OCI = 2 FEC, and CO′D = 2 CGD. Now the $\angle$ KDE = CDG. and DKE = DCG, $\therefore$ KED = CGD, $\therefore$ OCI = CO′H, and the right $\angle$ OIC = CHO′; $\therefore$ the $\triangle$s OCI, O′CH are equiangular;

$\therefore$ OC : O′C : : OI : CH; that is, OC : O′C : : KH : CH. Again, $AC^2 - CB^2 = AH^2 - BH^2 = (AH + HB)(AH - HB) = 2AK \cdot 2KH = 4\,AK \cdot KH$; but area of ABC = AK . CH, $\therefore$ four times area = 4 AK . CH; hence $AC^2 - CB^2$: 4 times area : : KH : CH; but KH : CH : : OC : O′C. Hence $AC^2 - CB$: 4 times area : : OC : O′C.

Lemma.—To construct a parallelogram, being given the diagonals and one of the angles.

Sol.—Let AB, CD be the diagonals, and E one of the angles.



On CD describe a segment CFD containing an $\angle$ equal to E. Bisect CD in G. With G as centre, and a radius equal to $\frac{1}{2}$ AB, describe a ⊙, cutting CFD in F. Join FG, and produce it to H. Cut off GH = GF. Join CF, DF, CH, DH. CFDH is the required parallelogram; for it has the $\angle$ CFD = E, and its diagonal FH = AB.

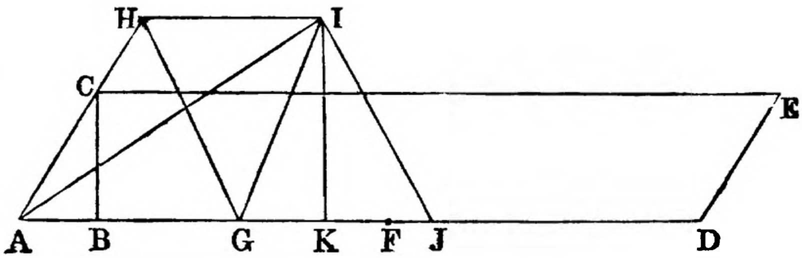
93. Let BAC be one of the $\angle$s, and AB the difference between its diagonals.

Sol.—Erect BC $\perp$ to AB; to AC apply a parallelogram ACED equal to four times the given area, and having BAC one of its angles. Bisect BD in F. Construct a ▭ AHIG, having one of its diagonals, AI = AF, and the other, HG = FD, and the $\angle$ BAC for one of its angles (*Lemma*). AHIG is the required parallelogram.

Dem.—Through I draw IJ $\parallel$ to HG, and let fall a $\perp$ IK on AD.

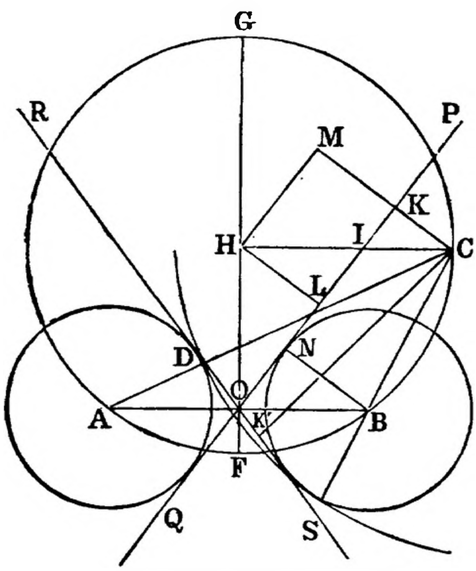
Now $AI^2 = AG^2 + GI^2 + 2AG \cdot GK$ (II. xii.), and (II. xiii.) $IJ^2 = JG^2 + GI^2 - 2JG \cdot GK = AG^2 + GI^2 - 2AG \cdot GK$; $\therefore AI^2 - IJ^2 = 4AG \cdot GK$. Again, AB = AF − FD, and AD = AF + FD,

$\therefore$ AB . AD = AF^2 − FD^2; but AF = AI, and FD = IJ, $\therefore$ AF^2 − FD^2 = AI^2 − IJ^2; $\therefore$ AB . AD = 4AG . GK. Again, since the $\triangle^s$ ABC, GKI are equiangular, we have AB : BC



: : GK : KI, $\therefore$ AB . AD : BC . AD : : 4AG . GK : 4AG . KI; hence BC . AD = 4AG . KI. Now BC . AD = ▭ AE, and 4AG . KI = 4 times ▭ AI, $\therefore$ ▭ AE = 4 times ▭ AI; but AE = 4 times the given area (const.). Hence AI is equal to the given area.

94. Let A, B be the centres of two equal ⊙^{s}, C the centre of a variable ⊙, which is touched externally by A in D, and



internally by B in E. Let O be the point of intersection of two transverse tangents PQ, RS. From C let fall $\perp^s$ CK, CK′ on PQ, RS. It is required to prove that OK . OK′ is constant.

Dem.—Join CB, and produce it to E. Join CA. Describe a ⊙ passing through the points C, A, B. Draw the diameter FG, passing through O, $\perp$ to AB. Let fall a $\perp$ CH on FG.

Produce CK, and draw HM ∥ to PQ. Let fall a ⊥ HL on PQ. Join BN, and let the sides BN, ON, OB of the △ ONB be denoted by a, b, c.

Now AC = AD + DC, and BC = CE − BE, ∴ AC − BC = 2AD, ∴ AD = $\frac{1}{2}$(AC − BC); that is, $a = \frac{1}{2}$(AC − BC); hence (IV. Ex. 16) a^2 = OF . GH; and since AB is bisected in O, AO = OB, and AO . OB = OF . OG, ∴ OB²; that is, e^2 = OF . OG, and ∴ ON²; that is, b^2 = OF . OH. Now since the △ˢ ONB, HMC are equiangular, and that HM = LK, we have $c : b :: \text{HC} : \text{LK}$;

$$\therefore\ \text{LK} = \frac{b\,.\,\text{HC}}{c}. \quad \text{In like manner OL} = \frac{a\,.\,\text{OH}}{c};$$

$$\therefore\ \text{OK} = \frac{b\,.\,\text{HC}}{c} + \frac{a\,.\,\text{OH}}{c}.$$

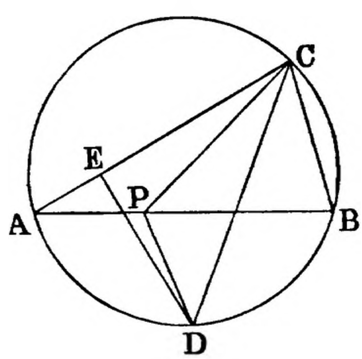
Similarly,
$$\text{OK}' = \frac{.\,\text{HC}}{c} - \frac{a\,.\,\text{OH}}{c};$$

$$\therefore\ \text{OK}\,.\,\text{OK}' = \frac{b^2\,.\,\text{HC}^2}{c} - \frac{a^2\,.\,\text{OH}^2}{c^2} = \frac{\text{OF}\,.\,\text{OH}\,.\,\text{FH}\,.\,\text{HG} - a^2\,.\,\text{OH}^2}{c^2}$$

$$= \frac{a^2\,.\,\text{OH}\,.\,\text{FH} - a^2\,.\,\text{OH}^2}{c^2} = \frac{a^2\,.\,\text{OH}\,(\text{FH} - \text{OH})}{c^2} = \frac{a^2 b^2}{c^2};$$

but a^2 is constant, since a is the radius of the circle, and c^2 is constant, because c is half the base of the △ ACB; ∴ $\frac{a^2 b^2}{c^2}$ is constant. Hence OK . OK′ is constant.

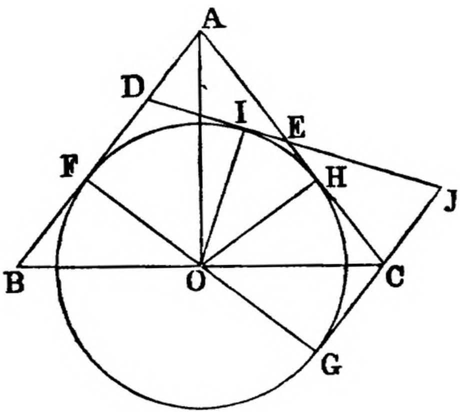
95. *Analysis.*—Let ABC be a △ whose base AB is given in magnitude and position, and vertical ∠ C is given in magnitude, and P the given point in AB, whose distance CP from the vertex



is equal to $\frac{1}{2}$(AC + CB). Describe a ⊙ about the △ ACB. Bisect the ∠ ACB by CD. Let fall a ⊥ DE on AB; then, because AB and the ∠ ACB are given, the ⊙ is given; and since

the $\angle$ ACB is bisected by CD, the arc AB is bisected in D; hence D is a given point. Again, because the $\angle$ ACB is given, its half, the $\angle$ DCE, is given, and the $\angle$ DEC is right; hence the $\triangle$ DCE is given in species, $\therefore$ the ratio of DC : CE is given; but CE = CP, because each is equal to $\frac{1}{2}$ (AC + CB); hence the ratio of DC : CP is given, and the points D, P are given. Hence the locus of the point C is a circle; and therefore the point C, where this locus cuts the $\odot$ ACB, is given.

96. Let O be the middle point of the base. F, H the points of contact of AB, AC with the $\odot$. Join OF, OH. Produce FO to

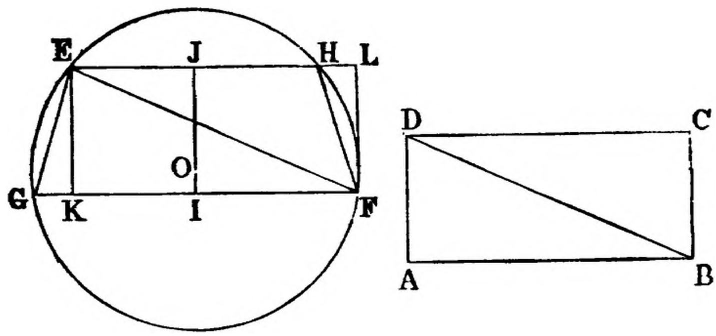


meet the $\odot$ in G. Join CG; then, since OC = OB, and OG = OF, and the $\angle$ COG = BOF, CG is equal to BF, and the $\angle$ OGC = OFB, and is therefore a right $\angle$; hence CG is a tangent. Again, because the $\angle$ AOC is right, and OH is $\perp$ to AC, AH . HC = OH^2; but AH = AF, and HC = CG; hence AF.CG = OH^2. In like manner, if I be the point of contact of DE with the circle, DF. JG = OI^2; but $OH^2 = OI^2$, $\therefore$ AF.CG = DF. JG; hence AF : DF : : JG : CG, $\therefore$ AD : DF : : JC : CG; $\therefore$ AD : JC : : DF : CG or FB; but, by similar $\triangle^s$, AD : JC : : AE : EC, $\therefore$ AE : EC : : DF : FB; hence, *componendo*, AC : CE : : DB : FB; hence AC . BF = BD . CE; but AC and BF are each given; $\therefore$ the rectangle BD . CE is given.

97. Let AB equal half the sum of the opposite sides, and the area equal the rectangle ABCD.

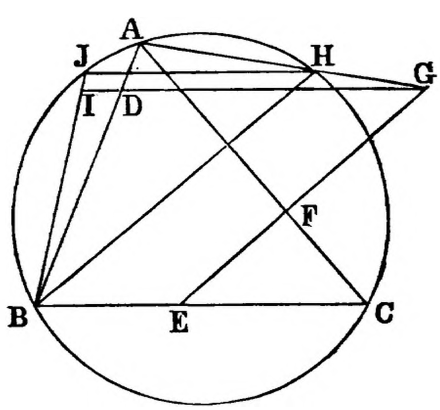
Sol.—Join BD, and in the $\odot$ place EF = BD. At the point F in EF make the $\angle$ EFG = ABD. Join EG, and draw EH || to FG. EHFG is the required trapezium.

Dem.—From the centre O let fall a ⊥ OI on FG, and produce it to meet EH in J. Let fall a ⊥ EK on FG. Produce EH, and draw FL parallel to EK. Because EF = BD, and the ∠ EFK



= DBA, and the right ∠ EKF = DAB, ∴ FK = AB, ∴ 2 AB = 2 IF + 2 IK; that is = FG + EH. Again, the ∠s EGF and EHF equal two right ∠s, and EHF, LHF equal two right ∠s; ∴ EGK = LHF, and the right ∠ EKG = HLF, and the side EK = FL; ∴ the Δs EGK, FLH are equal. To each add the figure EHFK, and EHFG = ELFK. Hence EHFG = ABCD.

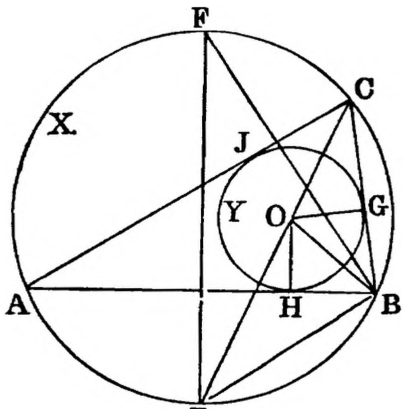
98. *Analysis.*—Let the polygon be the Δ ABC, whose sides pass through the points D, E, F. Join EF, and produce it.



Through B draw BH || to EF. Join AH, and produce it to meet EF in G. Now the ∠ GAC = HBC (III. xxi.), and HBC = GEC (I. xxix.), ∴ GEC = GAC; ∴ GAEC is a cyclic quadrilateral,

∴ EF . FG = AF . FC; but AF . FC is given; ∴ EF . FG is given, and EF is given; hence G is a given point. Join GD, and produce it. Through H draw HJ ∥ to GD. Join JB. Now the ∠ AHJ = ABJ, and AHJ = AGI, ∴ ABJ = AGI, ∴ AGBI is a cyclic quadrilateral; hence GD . DI = AD . DB, and is therefore given; but GD is given, ∴ DI is given, and I is a given point; and since JH, BH are respectively parallel to IG, EG, the ∠ JHB = IGE; but IGE is given, since the lines IG, EG are given in position, ∴ the ∠ JHB is given, ∴ the arc JB is given, ∴ the chord JB is given, and we have shown that I is a given point. Hence the question reduces to III. xv., Ex. 2. Similarly for a polygon of any number of sides.

99. Let the ⊙s X, Y be so related that the rectangle contained by the diameter of X, and the radius of Y, is equal to the rectangle contained by the segments of any chord of X passing



through the centre of Y; then, if from any point in the circumference of X we draw tangents CA, CB to Y, and join AB, it is required to prove that AB touches Y.

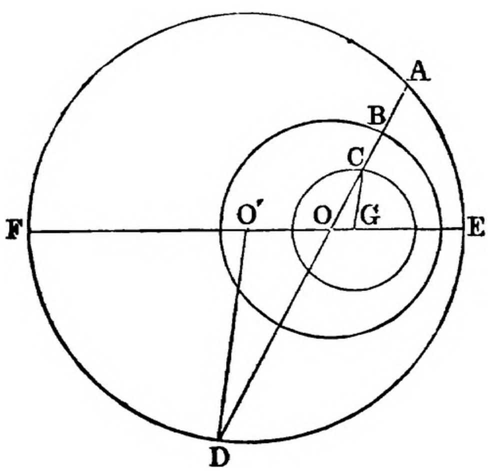
Dem.—Let O be the centre of Y. Join CO, and produce it to meet X in E. Through E draw EF the diameter of X. Join BE, BF, BO. Join O to G, the point of contact, and let fall a ⊥ OH on AB. Now the ∠ EFB = ECB (III. xxi.), and the right ∠ EBF = OGC, ∴ the △s EFB, OCG are equiangular; ∴ EF : EB : : OC : OG, ∴ EF . OG = EB . OC; but (hyp.) EF . OG = OC . OE, ∴ EB = OE, ∴ the ∠ EOB = EBO, ∴ EBO = OCB + OBC; that is, EBA + ABO = OCB + OBC; that is, ACE + ABO = OCB + OBC; but ACE = OCB; ∴ ABO = OBC, and the right ∠ OHB = OGB, and the side OB common, ∴ OH

= OG; but OG is the radius, ∴ OH is the radius; and hence AB touches Y. Similarly, wherever we take the point in the circumference of X, and draw tangents to Y, the base will touch Y.

Lemma.—If any point A is taken in the circumference of a ⊙, and A joined to O, the centre of another ⊙; and if we divide AO in C, so that OA . OC $= r^2$, r being the radius of O. It is required to prove that the locus of C is a ⊙.

Dem.—Suppose one ⊙ inside the other. Let O′ be the centre of the larger ⊙. Produce AO to meet O′ in D. Join DO′, OO′, and produce OO′ to meet O′ in E, F. Through C draw CG ∥ to DO′.

Now OA . OC $= r^2$, and OA . OD = OE . OF, ∴ OD : OC : : OE . OF : r^2; but the ratio OE . OF : r^2 is given, since r is the



radius of a given ⊙, and OE . OF is a given rectangle, ∴ the ratio OD : OC is given; and because the △s ODO′, OCG are equiangular, OD : OC : : OO′ : OG, ∴ the ratio OO′ : OG is given; but OO′ is given, ∴ OG is given; hence G is a given point. Again, OD : OC : : O′D : GC, ∴ the ratio O′D : GC is given; but O′D is given, since it is the radius of a given ⊙; ∴ GC is given, and we have shown that G is a given point. Hence the locus of C is a circle.

Def.—The point C is called the *inverse* of the point A, and the ⊙ through C the *inverse* of the ⊙ through A with respect to the ⊙ through B.

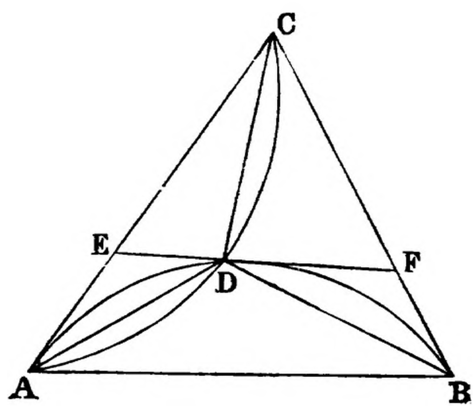
100. Let G, H, J be the points where Y touches the sides of the △ ABC. Join HG, GJ, JH. It is required to prove that the ⊙ inscribed in the △ GHJ touches a given circle.

Dem.—Join OA, OB, OC, cutting JH, HG, GJ in L, M, N. Then since L, M, N are the middle points of the sides of the △ GHJ, the ⊙ through these points will be the nine-points ⊙ of GHJ, and will (Ex. 31) touch its inscribed ⊙. Again, the ⊙ through LMN will evidently be the inverse of X with respect to Y (*Lemma*), and will be a given ⊙. Hence the inscribed ⊙ of the △ GHJ touches a given circle.

101. See "Sequel," Book VI., Prop. XII., Sect. iv., Cor. 2.

102. **Sol.**—Let A, B, C be the given points; join them, and on AB, AC describe segments of ⊙s containing ∠s equal to one-third of four right ∠s. Let them intersect in D. D is the point required.

Dem.—Join AD, BD, CD; and through D draw EF ⊥ to CD. Now the ∠ ADC = BDC, and EDC = FDC; ∴ ADE = BDF; hence ("Sequel," Book I., Prop. XXI., Cor. 1) the sum of AD



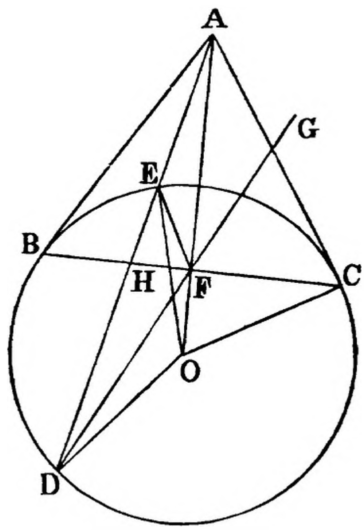
and DB is a minimum; and CD, being a ⊥, is less than any other line from C to EF. Hence the sum of the lines AD, BD, CD is a minimum.

103. Let AB, AC be the tangents, and O the centre. Join BC. Join AO, cutting BC in F. Through A draw AD, cutting the ⊙ in E, D, and BC in H. It is required to prove that AD is divided harmonically.

Dem.—Join OC, OD, OE, OF. Join DF, and produce it.

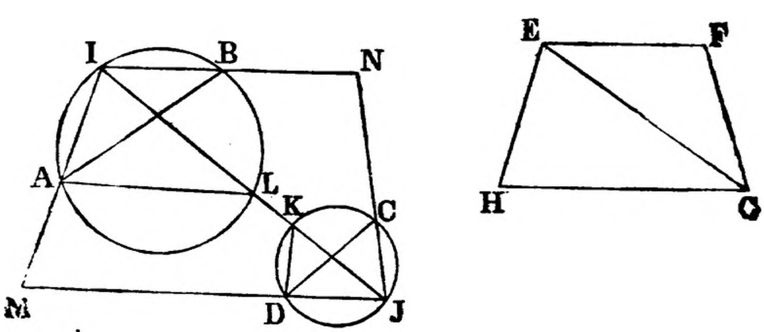
Now (I. XLVII., Ex. 1) AO . OF = OC^2 = OD^2; ∴ AO : OD :: OD : OF, and the ∠ AOD common; hence (VI.) the ∠ ADO

= OFD; but because OD = OE, the ∠ ODE = OED; ∴ OFD = OED; ∴ OFED is a cyclic quadrilateral; ∴ the ∠s EDO and EFO equal two right ∠s; but the ∠s EFO, EFA equal two right ∠s; ∴ EFA = EDO, and EDO = OFD; ∴ EFA = OFD, and AFB = OFB; ∴ DFH = AFH; hence the ∠ EFD is bisected internally, and the ∠ OFD = AFG, and OFD = EFA; ∴ EFA = AFG. Hence EFD is bisected externally, and therefore ED (III., Ex. 3) is divided harmonically in the points H, A.



104. Let A, B, C, D be the four points, and EFGH the given quadrilateral. It is required to construct a quadrilateral similar to EFGH whose sides shall pass through the points A, B, C, D.

Sol.—Join AB, and on it describe a segment AIB, containing an angle equal to FEH. Join CD, and on it describe a segment



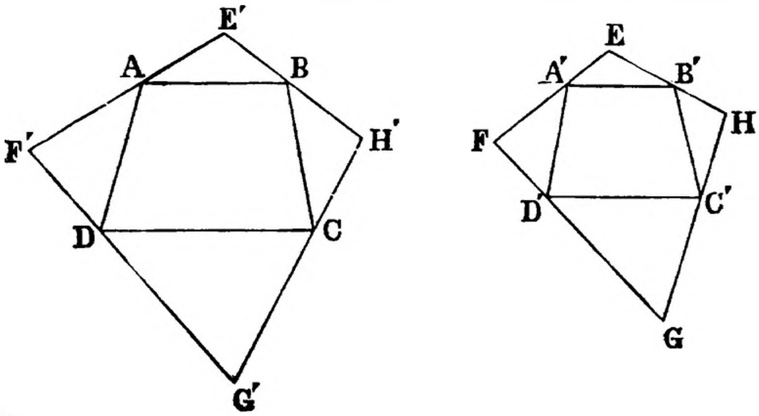
CJD, containing an angle equal to FGH. Join EG. At the point A in AB make the ∠ BAL = FEG; and at the point

D in DC make the ∠ CDK = EGF. Join KL, and produce it to meet the ⊙s in I, J. Join IA, IB, and produce. Join JC, JD, and produce. INJM is the required quadrilateral.

Dem.—For the ∠ BIL = BAL = FEG, and the ∠ CJK = CDK = EGF; ∴ the △s INJ, EFG are similar. And because the ∠ BIA = FEH, ∴ MIJ = HEG. Similarly, MJI = HGE, ∴ the △s MIJ, HEJ are similar. Hence the quadrilaterals are similar.

105. Let ABCD be the given quadrilateral, and EF, FG, GH, HE the given lines.

Sol.—Construct the quadrilateral E′F′G′H′ similar to EFGH, whose sides pass through the points A, B, C, D (103). Divide EF in A′, so that EA′ : A′F : : E′A : AF′; and divide EH in B′,



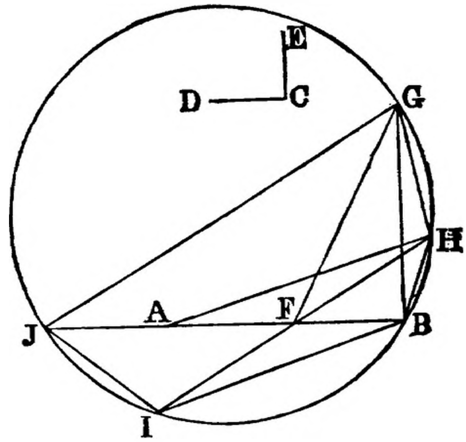
so that EB′ : B′H : : E′B : BH′; and similarly for the other sides. Join A′B′, B′C′, C′D′, D′A′. It is evident that A′B′C′D′ is similar to ABCD.

106. Let AB be the base, and DCE the difference of the base angles.

Sol.—Bisect AB in F. Draw BG, making the ∠ FBG = DCE, and the rectangle FB . BG equal to the rectangle under the sides. Join FG. Bisect the ∠ BFG by FH, and make FH a mean proportional between FG and FB. Join AH, BH. ABH is the required triangle.

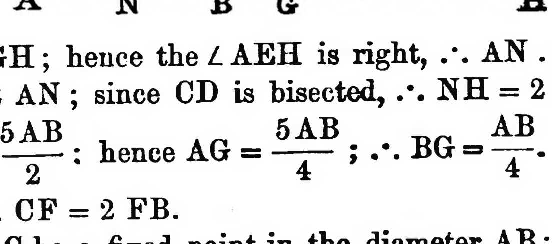
Dem.—Produce HF to I, so that IF = FH. Through G draw GJ ∥ to HI, and produce BA to meet it. Join IJ, IB, GH. Now (I. XXIX.) the ∠ HFB = GJF, and GFH = FGJ; but HFB = GFH (const.), ∴ GJF = FGJ, and ∴ FG = FJ. Now the ∠ GHI = JIH. To each add HGJ, and we have GHI + HGJ = JIH + HGJ; ∴ JIH + HGJ are equal to two right ∠s; hence HIJG is a

cyclic quadrilateral. And since FG . FB = FH^2 (const.), and FG = FJ, and FH^2 = FH . FI, ∴ FJ . FB = FM . FI; ∴ JIBH



is a cyclic quadrilateral. Hence the five points F, I, B, H, G are in a circle. Now the ∠ HBG = IBJ; but IBJ = BAH; ∴ HBG = BAH; ∴ FBG, that is DCE, is the difference between HAB and HBA. Again, the △s IBF, GBH are equiangular; ∴ IB : BF : : GB : BH; ∴ IB . BH = BF . BG; that is, AH . BH = BF . BG. This construction is due to Hamilton.

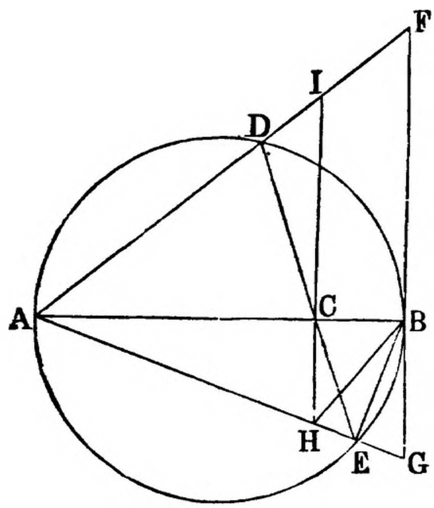
107. Let the line EF produced meet AB produced in G; cut off GH = EG. Join EH, and let fall the ⊥ EN. Now since (hyp.) the ∠ AEF = EAB, the △ AEG is isosceles; ∴ AG = EG,



and EG = GH; hence the ∠ AEH is right, ∴ AN . NH = EN^2; but EN = 2 AN; since CD is bisected, ∴ NH = 2 EN = 2 AB, ∴ AH = $\frac{5\,AB}{2}$; hence AG = $\frac{5\,AB}{4}$; ∴ BG = $\frac{AB}{4}$. Hence EC = 2 BG; ∴ CF = 2 FB.

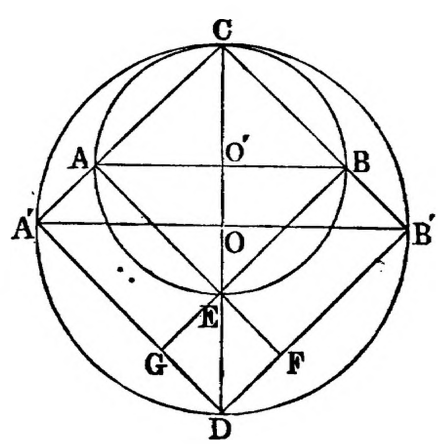
108. Let C be a fixed point in the diameter AB; DE a chord passing through C. Join AD, AE. At B draw FG a tangent to the ⊙, and produce AD, AE to meet it in F and G. It is required to prove that BF . BG is constant.

Dem.—Through C draw HI ∥ to BG, meeting AF, AG in I, H. Join BE, BH. Now the ∠ BCH is right, and BEH is right, ∴ CBEH is a cyclic quadrilateral, ∴ the ∠ BEC = BHC; but BEC = BAD (III. XXI.), ∴ BHC = BAD, ∴ the four points B, H, A, I are concyclic; hence IC . CH = AC . CB; and because the $\triangle^s$ ACI, ABF are equiangular, AC : AB : : IC : BF; and since the $\triangle^s$ ACH, ABG are equiangular, we have AC : AB : : CH : BG, ∴ $AC^2 : AB^2$: : IC . CH : BF . BG;



that is, $AC^2 : AB^2$: : AC . CB : BF . BG; but the first three terms of this proportion are constant. Hence the fourth, BF . BG, is constant.

109. Let O, O′ be the centres of the ⊙s, and C the point of contact.

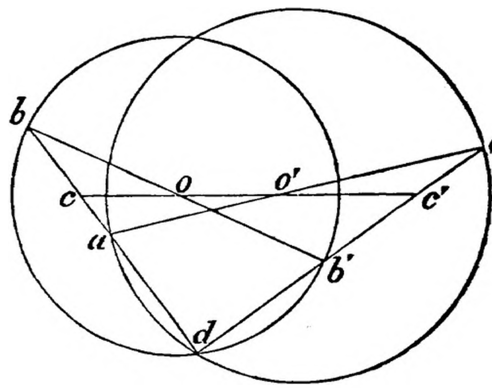


Dem.—Join OO′, and produce it; OO′ must pass through C.

Let E, D be the other points in which it meets the ⊙s. Join AD, BD, AE, BE, and let AE, BE meet B′D, A′D in F, G. Now each of the ∠s CBE, CB′D is right (III. xxxi.); ∴ BG is ∥ to B′D, and BB′ = GD. In like manner AA′ = GE, ∴ $AA'^2 + BB'^2 = GE^2 + GD^2 = DE^2$; but DE^2 is constant, since it is equal to the square of the difference of the diameters. Hence $AA'^2 + BB'^2$ is constant.

110. Let d be the point of intersection.

Dem.—Join aa', bb'; those lines must pass respectively through



the centres o', o (hyp.). Now the sides of the △ bdb' are cut by cc' in the points c, o, c'; hence (vi., Ex. 5),

$$\frac{dc}{cb}\cdot\frac{bo}{ob'}\cdot\frac{b'c'}{c'd} = 1;\ \text{but } bo = ob';\ \therefore\ \frac{dc}{cb}\cdot\frac{b'c'}{c'd} = 1;\ \therefore\ \frac{dc}{cb} = \frac{c'd}{b'c'}.$$

In like manner, from the △ ada', we get

$$\frac{dc}{ca} = \frac{dc'}{a'c'},\ \therefore\ \frac{ca}{cb} = \frac{a'c'}{b'c'};$$

that is, $ac : cb :: a'c' : b'c'$. Hence $ab : cb :: a'b' : b'c'$.

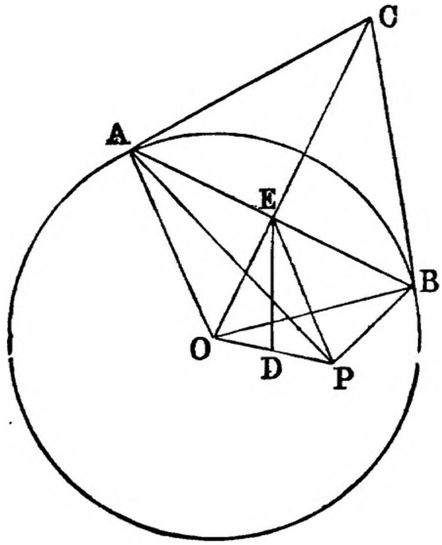
111. "Sequel," Diagram, p. 32. By "Sequel," Prop. viii., Cor. 3, p. 32, we have AB . QR = EP². Similarly AB, multiplied by the diameter of the ⊙ touching EP, the semicircle ACB, and the semicircle on AP as diameter, is equal to EP². Hence the ⊙s are equal.

112. Let P be the given point, AB the chord, and CA, CB the tangents.

Dem.—Let O be the centre. Join OA, OB, OP, OC, PE. Bisect OP in D. Join DE.

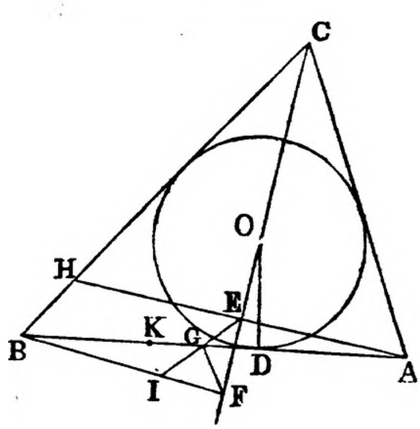
Now because OA = OB, OC common, and the base CA = CB, the ∠ AOC = BOC; and since AO = BO, OE common, and

the $\angle$ AOE = BOE, the base AE = BE. Now $AO^2 = AE^2 + EO^2$; but (I. XII., Ex. 2) the lines AE, EB, EP are equal, $\therefore AO^2 = OE^2 + EP^2 = 2\,OD^2 + 2\,DE^2$ (II. X., Ex. 2);



but AO^2 is given, $\therefore 2\,OD^2 + 2\,DE^2$ is given, and $2\,OD^2$ is given, since OP is given, $\therefore$ DE is given, and D is a fixed point. Hence the locus of E is a $\odot$, having D as centre, and DE as radius. Now (I. XLVII., Ex. 1) $CO \,.\, OE = OA^2 = R^2$; $\therefore$ C, E are inverse points with respect to the $\odot$ ABF, and it has been shown that the locus of E is a $\odot$. Hence the locus of C is a circle.

113. Let ABC be a $\triangle$, O the centre of the inscribed $\odot$, and



D the point in which the $\odot$ touches AB. Join CO, and produce

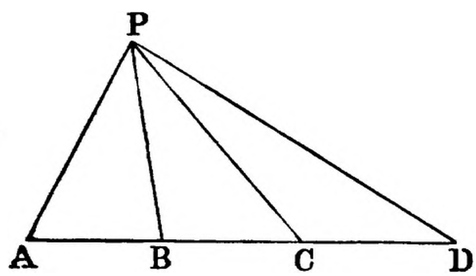
it. CO is the bisector of the ∠ ACB. From A let fall a ⊥ AE on CF, and produce it to meet CB in H, and from B let fall a ⊥ BF on CF. It is required to prove that AD . DB = AE . BF.

Dem.—Join OD. Bisect AB in G. Join EG, and produce it to meet BF in I. Join FG. Let the sides of the △ ABC be denoted by a, b, c, and we have (IV. iv., Ex. 2) BD = $(s - b)$, AD = $(s - a)$, ∴ BD − AD; that is, 2 GD = $(a - b)$. Now since the ∠ ECH = ECA, and the right ∠ CEH = CEA, and the side EC common, ∴ CH = CA, ∴ CB − CA = BH; that is $(a - b)$ or 2 GD = BH = 2 GE; ∴ GD = GE. In like manner GD = GF, and GD = GI; hence the lines GE, GD, GF, GI are equal, and the ⊙, with G as centre, and GD as radius, will pass through E, F, I. Let it cut BD in K. Now (III. xxxvi.) BD . BK = BF . BI. But since AG = GB, and DG = GK, AD = KB. Also BI = HE = AE. Hence BD . AD = BF . AB.

114. See "Sequel," Book VI., Prop. x., Sect. i., Cor. 1.

115, See "Sequel," Book VI., Prop. x., Sect. i., Cor. 2.

116. *Analysis.*—Let P be the required point. Join AP, BP, CP, DP. Now (hyp.) the ∠ APC is bisected, ∴ AB : BC :: AP : CP (iii.); but the ratio AB : BC is given, ∴ AP : CP



is given, and the base AC is given; hence (iii., Ex. 6) the locus of P is a circle. Similarly for the △ BPD, the locus of P is another ⊙. Hence the point in which these ⊙s intersect is the point required.

117. Let ABC be a △ whose sides are denoted by a, b, c. Bisect the ∠ ACB by CD, and let CD be denoted by λ. Now (iii.) we have $a : b ::$ BD : DA; ∴ $(a + b) : b ::$ BA : AD; that is $(a + b) : b :: c :$ AD; ∴ AD $= \dfrac{bc}{a + b}$. Similarly, BD $= \dfrac{ac}{a + b}$,

∴ BD . DA $= \dfrac{abc^2}{(a + b^2)}$; but $ab =$ BD . DA + CD^2 (xvii., Ex, 1);

$\therefore ab = \dfrac{abc^2}{(a+b)^2} + \text{CD}^2$; that is, $ab - \dfrac{abc^2}{(a+b)^2} = \text{CD}^2$; that is,

$$ab\left\{1 - \frac{c^2}{(a+b)^2}\right\} = \text{CD}^2; \text{ hence } \gamma^2 = ab\left\{\frac{(a+b^2) - c^2}{(a+b)^2}\right\}$$

$$= \frac{ab\,(a+b+c)\,(a+b-c)}{(a+b)^2} = \frac{4\,ab\,.\,s\,.\,s-c}{(a+b)^2}.$$

In like manner, denoting the bisectors of the angles A, B by α. β respectively, we have

$$\alpha^2 = \frac{4\,bc\,.\,s\,.\,s-a}{(b+c)^2}, \text{ and } \beta^2 = \frac{4\,ca\,.\,s\,.\,s-b}{(c+a)^2};$$

hence

$$\alpha^2\beta^2\gamma^2 = \frac{64\,a^2b^2c^2\,.\,s^2\,(s\,.\,s-a\,.\,s-b\,.\,s-c)}{(a+b)^2\,(b+c)^2\,(c+a)^2} = \frac{64\,a^2b^2c^2\,.\,s^2.(\text{area})^2}{(a+b)^2(b+c)^2(c+a)^2}.$$

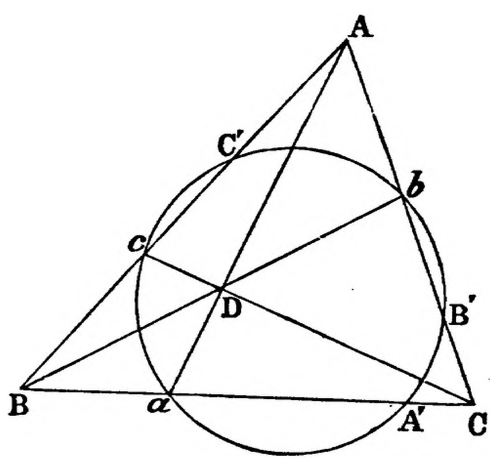
Hence, $$\alpha\beta\gamma = \frac{8\,abc\,.\,s\,.\,\text{area}}{(a+b)\,(b+c)\,(c+a)}.$$

118. Let Aa', Bb', Cc' be the bisectors of the $\angle$s; then (III.) we have $c : a :: \text{A}b' : b'\text{C}$; $\therefore c : c+a :: \text{A}b' : b$; $\therefore \text{A}b' = \dfrac{bc}{c+a}$.

In like manner, $\text{B}c' = \dfrac{ca}{a+b}$, and $\text{C}a' = \dfrac{ab}{b+c}$; $\therefore \text{A}b'.\text{B}c'.\text{C}a'$

$$= \frac{a^2b^2c^2}{(a+b)\,(b+c)\,(c+a)}.$$

119. Let ABC be a Δ. Draw any three lines Aa, Bb, Cc,



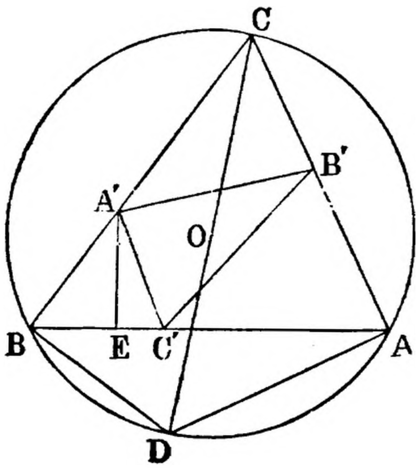
intersecting in D. Describe a $\odot$, passing through the points

a, b, c, and cutting the sides of the $\triangle$ ABC in A′, B′, C′. It is required to prove that the lines AA′, BB′, CC′ are concurrent.

Dem.—Now we have $Ab \cdot AB' = Ac \cdot AC'$, $Bc \cdot BC' = Ba \cdot BA'$, and $Ca \cdot CA' = Cb \cdot CB'$; $\therefore$ $(Ab \cdot Bc \cdot Ca)$ $(AB' \cdot BC' \cdot CA') =$ $(aB \cdot bC \cdot cA)$ $(A'B \cdot B'C \cdot C'A)$; but $Ab \cdot Bc \cdot Ca = aB \cdot bC \cdot cA$ (Ex. 4); $\therefore AB' \cdot BC' \cdot CA' = A'B \cdot B'C \cdot C'A$. And hence the lines AA′, BB′, CC′ are concurrent.

120. **Dem.**—Describe a ⊙ about ABC. Let O be the centre. Join CO, and produce it to meet the circumference in D. Join DA, DB, and from A′ let fall a ⊥ A′E on AB.

Now if we denote the sides by a, b, c, and the parts A′B, B′C, C′A, by x, y, z, we have $(a - x)$ $(b - y)$ $(c - z) = AB' \cdot BC' \cdot CA'$, and $xyz = A'B \cdot B'C \cdot C'A$; $\therefore abc - (abz + bcx + cay) + ayz + bzx + cxy = AB' \cdot BC' \cdot CA' + A'B \cdot B'C \cdot C'A$. Again, since the $\triangle$s

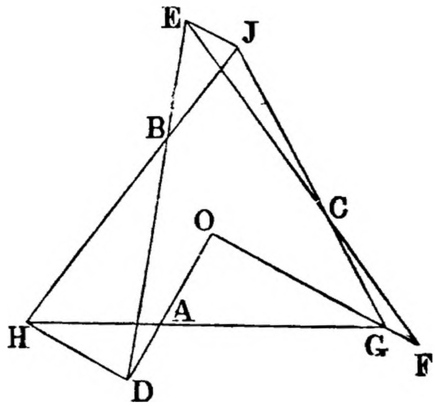


BA′E, ACD are equiangular, we have $BA' : A'E :: CD : CA$; that is (denoting CD by δ), $x : A'E :: \delta : b$; $\therefore bx = \delta \cdot A'E$; $\therefore bx \cdot BC' = \delta \cdot A'E \cdot BC' = \delta \cdot 2\,A'BC'$; that is, $bx\ (c - z) =$ $\delta \cdot 2\,A'BC'$; $\therefore (bcx - bxz) = \delta \cdot 2\,A'BC'$. In like manner $(cay - cyx)$ $= \delta \cdot 2\,B'CA'$, and $(abz - axy) = \delta \cdot 2\,C'AB'$, and "Sequel," Book VI., Prop. v., Sect. i.) $abc = \delta \cdot 2\,ABC$; $\therefore abc - (bcx + cay + abz) + (ayz + bzx + cxy) = \delta \cdot 2\,A'B'C'$. Hence $AB' \cdot BC' \cdot CA' + A'B \cdot B'C \cdot C'A = \delta\, 2\,A'B'C'$.

121. Let A, B, C be the fixed points, and the given ratio that of 2 : 1.

Sol.—Take any point O. Join OA, and produce it to D, so that OA = 2 AD. Join DB, and produce to E until DB = 2 BE. Join

EC, and produce it to F, so that EC = 2 CF. Join OF, and divide it in G, so that OG = 8 FG. Join GA, and produce it; and through D draw DH ∥ to OG. Join HB, and produce it; and



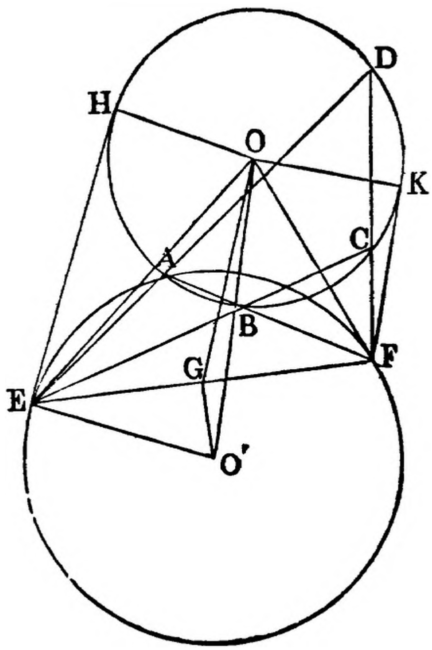
through E draw EJ ∥ to DH. Join JC, GC. GHJ is the required △.

Dem.—The △s OAG, DAH are equiangular; ∴ OA : AD :: OG : DH; ∴ OG = 2 DH; but OG = 8 GF; ∴ DH = 4 GF. Similarly, from the △s BDH, BEJ we get DH = 2 JE; ∴ JE = 2 GF, and EC = 2 CF (const.), and the ∠ JEC = GFC; hence (VI.) th ∠ JCE = GCF, and therefore JC and GC are in the same straight line; and evidently the sides are divided in the points A, B, C in the given ratio. Similarly for any polygon of an odd number of sides, and for any given ratio.

122. Let ABCD be a cyclic quadrilateral whose third diagonal EF is a chord of another given ⊙. Bisect EF in G. It is required to prove that the locus of G is a circle.

Dem.—Let O, O′ be the centres. Join OG, O′G, O′E. From E, F draw tangents EH, FK to O. Join OH, OK, EO, FO, OO′.

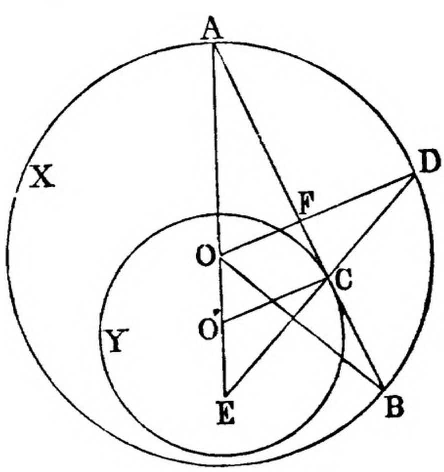
Now $4\,EG^2 + 4\,GO'^2 = 4\,EO'^2$; that is, $EF^2 + 4\,GO'^2 = 4\,EO'^2$; but $EF^2 = EH^2 + FK^2$ (III., Ex. 19), and $OH^2 + OK^2 = 2\,OH^2$. Adding, we get $EO^2 + OF^2 + 4\,GO'^2 = 4\,EO^2 + 2\,OH^2$; that is (II. x., Ex. 2), $2\,EG^2 + 2\,GO^2 + 4\,GO'^2 = 4\,EO'^2 + 2\,OH^2$, and $2\,EG^2 + 2\,GO'^2 = 2\,EO'^2$. Subtracting, we have $2\,GO^2 + 2\,GO'^2 = 2\,EO'^2 + 2\,OH^2$; ∴ $GO^2 + GO'^2 = EO'^2 + OH^2$; but EO'^2 and OH^2 are given; ∴ $GO^2 + GO'^2$ is given. Therefore OGO′ is a △ whose base is given, and the sum of the



squares of its sides. Hence (II. x., Ex. 3) the locus of G is a circle.

123. Let X, Y be the ⊙s, and let AB, a chord of X, touch Y in C. Bisect the arc AB in D. Join DC. It is required to prove that DC passes through a given point.

Dem.—Join OO′, and produce OO′, DC to meet in E. E is the given point. Join OA, OB. Now OA = OB, OF common,



and the ∠ AOF = BOF ; hence the ∠ AFO = BFO ; ∴ the ∠ AFO is right, and FCO′ is right ; ∴ OD is ∥ to O′C ; hence

(II.) the $\triangle$s DOE, CO′E are equiangular; $\therefore$ DO : CO′ :: OE : O′E; hence the ratio OE : O′E is given, $\therefore$ the ratio OO′ : O′E is given; but OO′ is given, $\therefore$ O′E is given, and O′ is a given point; $\therefore$ E′ is a given point.

124. Let ABC be a given triangle. From a point P, within it, let fall $\perp$s PD, PE, PF on the sides BC, CA, AB. Join DE, EF, FD, and let the area of DEF be given. It is required to prove that the locus of P is a circle.

Dem.—Join AP, BP, CP. Because each of the $\angle$s AEP, AFP is right, AFPE is a cyclic quadrilateral. Bisect AP in G. G is the centre of the $\odot$. Similarly, BDPF, CDPE are cyclic quadrilaterals; and H, J, the middle points of BP, CP, are the centres of their circumscribed $\odot$s. Join DH, HF, FG, GE, EJ, JD. Produce FG, and let fall a $\perp$ EK on it. Because AG = GP, the $\triangle$ AGF = PGF; $\therefore$ AFP = 2 PGF. In like manner, AEP = 2 EGP; hence the quadrilateral AFEP = 2 EGFP. Similarly, BFPD = 2 FHDP, and CDPE = 2 PDJE; hence the area of the figure EGFHDJ is given; but the area of FDE is given (hyp.); hence the sums of the areas EGF, FHD, DJE is given. Again, the $\angle$ FGE = 2 FAE (III. xx), $\therefore$ the $\angle$ FGE is given; $\therefore$ the $\angle$ KGE is given, and the $\angle$ GKE is right; hence the $\triangle$ EGK is given in species; $\therefore$ the ratio EG : EK is given; $\therefore$ the ratio EG . FG : EK . FG is given; but EK . FG = 2 $\triangle$ EGF, and EG . FG = FG^2; $\therefore$ the $\triangle$ EGF has a given ratio to FG^2, and FG^2 has a given ratio to AP^2, since AP = 2 FG; $\therefore \dfrac{\text{EGF}}{\text{AP}^2}$ is given. Suppose it equal to l; hence EGF = $l \, . \, \text{AP}^2$. In like manner, FHD = $m \, . \, \text{BP}^2$, and DJE = $n \, . \, \text{CP}^2$; but we have shown that the sum of EGF, FHD, DJE is given; hence $l \, . \, \text{AP}^2 + m \, . \, \text{BP}^2 + n \, . \, \text{CP}^2$ is given. And hence (*Lemma* to Ex. 60) the locus of P is a circle. Similarly, the proposition may be proved for a figure of any number of sides.

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